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Extreme values of the sum of squares of degrees of bipartite graphs

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Abstract

In this paper we determine the minimum and maximum values of the sum of squares of degrees of bipartite graphs with a given number of vertices and edges.

Keywords: Degree squares; Bipartite graphs; Extreme values

1. Introduction

All graphs considered in this paper are finite, undirected and simple. For terminology and notation not defined here we follow those in Bondy and Murty [4].

In this paper we study an extremal problem on the degree sequences of bipartite graphs: determine the minimum and maximum values of the sum of squares of degrees of bipartite graphs with a given number of vertices and edges.

Related problems for general graphs have been studied in [2,3,13]. It is easy to see that, among all the graphs with a given number of vertices and edges, a graph has the minimum sum of squares of degrees if and only if its maximum degree exceeds its minimum degree by at most one. Since such graphs always exist, the problem for general graphs is trivial in the minimum case. However, the problem is much complicated in the maximum case. Ahlswede and Katona [2] gave a solution for this problem. At the same time, for the family of bipartite graphs with n vertices, m edges and one partite side of size k, they determined a bipartite graph such that the sum of squares of its degrees is maximum. Boesch et al. [3] studied a more advanced problem for the maximum case: among all the graphs with a given number of vertices and edges, find the ones where the sum of squares of degrees is maximum. It was showed that every such graph is a threshold graph (see the definition in [10]). They constructed two threshold graphs and proved that at least one of them has the maximum sum of squares of degrees among graphs with a given number of vertices and edges. Peled et al. [13] further studied this problem and showed that, among all the graphs with a given number of vertices and edges, if a graph has the maximum sum of squares of degrees, then it must belong to one of the

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Table 1
Degrees of the vertices in the graph $B^s(n, m)$

$d(x_i)$	i	$d(y_j)$	j
n is even			
$t+1$	$1 \le i \le \frac{r}{2}$	$t+1$	$1 \le j \le \frac{r}{2}$
t	$\frac{r}{2}+1 \le i \le \frac{n}{2}$	t	$\frac{r}{2}+1 \le j \le \frac{n}{2}$
n is odd and $nt \le 2m < nt+t$			
$t+1$	$1 \le i \le \frac{r+t}{2}$	t	$1 \le j \le \frac{n+r-t+1}{2}$
t	$\frac{r+t}{2}+1 \le i \le \frac{n-1}{2}$	$t-1$	$j=0$ or $\frac{n+r-t+1}{2}+1 \le j \le \frac{n-1}{2}$
n is odd and $nt+t \le 2m \le nt+n-t-1$			
$t+1$	$1 \le i \le \frac{r+t}{2}$	$t+1$	$1 \le j \le \frac{r-t}{2}$
t	$\frac{r+t}{2}+1 \le i \le \frac{n-1}{2}$	t	$j=0$ or $\frac{r-t}{2}+1 \le j \le \frac{n-1}{2}$
n is odd and $nt+n-t+1 \le 2m < nt+n$			
$t+2$	$1 \le i \le \frac{r+t-n+1}{2}$	$t+1$	$1 \le j \le \frac{r-t}{2}$
$t+1$	$\frac{r+t-n+1}{2}+1 \le i \le \frac{n-1}{2}$	t	$j=0$ or $\frac{r-t}{2}+1 \le j \le \frac{n-1}{2}$

six particular classes of threshold graphs. Other types of bounds for the sum of squares of degrees of general graphs can be found in the literature, i.e., see [1,5–9,11,12,14].

The rest of the paper is organized as follows. In Section 2 we present some notation and lemmas that will be used later. The minimum and maximum sums of squares of degrees of bipartite graphs with a given number of vertices and edges are presented in Sections 3 and 4, respectively.

2. Notation and lemmas

Let x be a real number. We use $\lfloor x \rfloor$ to represent the largest integer not greater than x and $\lceil x \rceil$ to represent the smallest integer not less than x. The sign of x, denoted by $\mathrm{sgn}(x)$, is defined as 1, -1, and 0 when x is positive, negative and zero, respectively.

We use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and maximum degree of a graph G, respectively. By $n_i(G)$ we denote the number of vertices in G with degree i. If S is a set of vertices, we use $\delta(S)$ and $\Delta(S)$ to denote the minimum degree and the maximum degree of the vertices in S, respectively. Let S^i represent the set of vertices in S with degree i.

Let n, m and k be three positive integers. We use $B(n, m)$ to denote a bipartite graph with n vertices and m edges, and $B(n, m, k)$ to denote a $B(n, m)$ with a bipartition (X, Y) such that $|X| = k$. By $\mathcal{B}(n, m, k)$ we denote the set of graphs of the form $B(n, m, k)$.

Let $n \ge 2$ be an even integer and $t \le n/2$ a nonnegative integer. By $B_{n,t}$ we denote the bipartite graph with vertices $x_1, x_2, \ldots, x_{n/2}, y_1, y_2, \ldots, y_{n/2}$ and edges $x_i y_j$ with $i < j \le i + t$ (where the addition is taken modulo $n/2$) for $i, j = 1, 2, \ldots, n/2$.

For two integers n and m with $n \ge 2$ and $0 \le m \le \lfloor n/2 \rfloor \lceil n/2 \rceil$, let $2m = nt + r$, where $0 \le r < n$. We define a bipartite graph $B^s(n, m)$ with n vertices and m edges as follows.

Case 1. n is even. Define $B^s(n, m) = B_{n,t} \cup \{x_i y_i | 1 \le i \le r/2\}$.

Case 2. n is odd and $nt \le 2m < nt + t$. Define $B^s(n, m) = B^s(n-1, m-t+1) \cup \{x_i y_0 | (n+r-t+1)/2 + 1 \le i \le (n+r+t-1)/2\}$, where the addition is taken modulo $(n-1)/2$.

Case 3. n is odd and $nt + t \le 2m \le nt + n - t - 1$ or $nt + n - t + 1 \le 2m < nt + n$. Define $B^s(n, m) = B^s(n-1, m-t) \cup \{x_i y_0 | (r-t)/2 + 1 \le i \le (r+t)/2\}$, where the addition is taken modulo $(n-1)/2$.

The degrees of the vertices of the graph $B^s(n, m)$ are shown in Table 1.

If n is odd and $nt \le 2m < nt + t$ or $nt + n - t + 1 \le 2m < nt + n$, then from the above table, it can be checked that $n_{\delta(B^s(n,m))} = \frac{(n+1)(\delta+1)}{2} - m$, $n_{\delta(B^s(n,m))+1} = n - \delta - 1$ and $n_{\delta(B^s(n,m))+2} = m - \frac{(n-1)(\delta+1)}{2}$, where

$$\delta = \begin{cases} t-1, & \text{if } n \text{ is odd and } nt \le 2m < nt+t; \\ t, & \text{if } n \text{ is odd and } nt+n-t+1 \le 2m < nt+n. \end{cases}$$

Suppose that n, m and k are three integers with $n \geq 2$, $0 \leq m \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$ and $\lceil n/2 \rceil \leq k \leq n-1$. Let $m = qk + r$, where $0 \leq r < k$. Then $B^l(n, m, k)$ is defined as a bipartite graph in $\mathcal{B}(n, m, k)$ such that q vertices in Y are adjacent to all the vertices of X and one more vertex in Y is adjacent to r vertices in X. We use $B^l(n, m)$ to denote a graph $B^l(n, m, k_0)$ with $k_0 = \max\{k | m = qk + r, 0 \leq r < k, \lceil n/2 \rceil \leq k \leq n - q - \mathrm{sgn}(r)\}$.

Let $D = (d_1, d_2, \ldots, d_n)$ be a nonnegative integer sequence. Define $\sigma_2(D) = \sum_{i=1}^n d_i^2$. If D is the degree sequence of a graph G, then we define $\sigma_2(G) = \sigma_2(D)$.

The following lemma is obvious.

Lemma 1. *Let m be a nonnegative integer and $D = (d_1, d_2, \ldots, d_n)$ an integer sequence with $0 \leq d_i \leq n-1$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n d_i = 2m$. Then*

$$\sigma_2(D) \geq \left(4m - n - n\left\lfloor \frac{2m}{n} \right\rfloor\right)\left\lfloor \frac{2m}{n} \right\rfloor + 2m,$$

and the equality holds if and only if $|d_i - d_j| \leq 1$ for $1 \leq i < j \leq n$.

Lemma 2 ([2]). *Let m, n and k be three integers with $n \geq 2$, $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and $\lceil n/2 \rceil \leq k \leq n-1$. Suppose $m = qk + r$, where $0 \leq r < k$. Then $\sigma_2(B^l(n, m, k))$ attains the maximum value among all the graphs in $\mathcal{B}(n, m, k)$.*

3. Minimum value of the sum of squares of degrees

Theorem 1. *Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$. Then $\sigma_2(B^s(n, m))$ attains the minimum value among all the bipartite graphs with n vertices and m edges.*

Proof. Let $2m = nt + r$, where $0 \leq r < n$. If n is even, or n is odd and $nt + t \leq 2m \leq nt + n - t - 1$, then from Table 1 we have $\Delta(B^s(n, m)) - \delta(B^s(n, m)) \leq 1$. It follows from Lemma 1 that $\sigma_2(B^s(n, m))$ attains the minimum value in these cases.

If n is odd, then $nt + n - t = (n-1)t + n$ is odd too. So we have $2m \neq nt + n - t$. Note that if n is odd and $m < n$, then $nt + t \leq 2m \leq nt + n - t - 1$. So in the following we need only consider the case where n is odd, $m \geq n$, and $nt \leq 2m < nt + t$ or $nt + n - t + 1 \leq 2m < nt + n$.

Suppose that G is a bipartite graph such that $\sigma_2(G)$ attains the minimum value among all the bipartite graphs with n vertices and m edges.

Claim 1. $\delta(G) \geq 1$.

Proof. Since $m \geq n$, there must be one vertex u with $d(u) \geq 2$. If $\delta(G) = 0$, let v be a vertex with $d(v) = 0$. Choose one neighbor w of u and set $G' = G - uw + vw$. Clearly G' is still a bipartite graph. Then

$$\sigma_2(G') - \sigma_2(G) = 2(1 - d(u)) < 0,$$

a contradiction. □

Let (X, Y) be the bipartition of G. By the symmetry of X and Y, we assume that $|X| < |Y|$.

Claim 2. $\Delta(X) - \delta(X) \leq 1$ *and* $\Delta(Y) - \delta(Y) \leq 1$.

Proof. We only prove $\Delta(X) - \delta(X) \leq 1$. The other assertion can be proved similarly.

We prove this by contradiction. Suppose that there exist two vertices x and x' in X such that $d(x) = \Delta(X)$, $d(x') = \delta(X)$ and $d(x) - d(x') > 1$. Then there must be one vertex $y \in Y$ such that $xy \in E(G)$ but $x'y \notin E(G)$. Set $G' = G - xy + x'y$. Clearly G' is still a bipartite graph. So we have

$$\begin{aligned}\sigma_2(G') - \sigma_2(G) &= (d(x) - 1)^2 + (d(x') + 1)^2 - d(x)^2 - d(x')^2 \\ &= 2(d(x') - d(x) + 1) \\ &< 0,\end{aligned}$$

a contradiction. □

Claim 3. $\Delta(X) = \Delta(G)$ *and* $\delta(Y) = \delta(G)$.

Proof. Clearly, $\Delta(G)-\delta(G)\ge 1$. We distinguish two cases.

Case 1. $\Delta(G)-\delta(G)=1$.

Suppose $\Delta(X)\neq\Delta(G)$. Then $\Delta(X)=\delta(X)=\delta(G)$. So we have

$$\sum_{x\in X} d(x)=|X|\delta(G)<|Y|\delta(G)\le\sum_{y\in Y} d(y),$$

a contradiction. Suppose $\delta(Y)\neq\delta(G)$. Then $\Delta(Y)=\delta(Y)=\Delta(G)$. So we have

$$\sum_{y\in Y} d(y)=|Y|\Delta(G)>|X|\Delta(G)\ge\sum_{x\in X} d(x),$$

again a contradiction.

Case 2. $\Delta(G)-\delta(G)\ge 2$.

Suppose $\Delta(X)\neq\Delta(G)$. Then $\Delta(Y)=\Delta(G)$. By Claim 2, we have

$$\sum_{y\in Y} d(y)>|Y|(\Delta(G)-1)>|X|(\Delta(G)-1)\ge\sum_{x\in X} d(x),$$

a contradiction. The result $\delta(Y)=\delta(G)$ follows from Claim 2 immediately. $\square$

For simplicity, in the following we use Δ and δ instead of $\Delta(G)$ and $\delta(G)$, respectively.

Claim 4. $\Delta-\delta=2$.

Proof. Since $|X|<|Y|$, G cannot be a regular bipartite graph. So we have $\Delta-\delta\neq 0$.

Suppose $\Delta-\delta\ge 3$. If $|X^{\Delta-1}|=0$ or $|Y^{\delta+1}|=0$, then by Claims 1 and 2, there exist two vertices $x^*\in X^{\Delta}$ and $y^*\in Y^{\delta}$ such that $x^*y^*\in E(G)$. If $|X^{\Delta-1}|\neq 0$ and $|Y^{\delta+1}|\neq 0$, and there exist no edges connecting vertices in X^{Δ} and vertices in Y^{δ}, then by Claims 1 and 2, we can choose three vertices $x^*\in X^{\Delta}$, $x'\in X^{\Delta-1}$ and $y^*\in Y^{\delta}$ such that $x'y^*\in E(G)$ but $x^*y^*\notin E(G)$. Since $Y^{\delta+1}$ contains all the neighbors of x^* and $d(x^*)=d(x')+1$, there must exist one vertex $y'\in Y^{\delta+1}$ such that $x^*y'\in E(G)$ but $x'y'\notin E(G)$. Set $G'=G-x^*y'-x'y^*+x^*y^*+x'y'$. Then G' has the same degree sequence as G and there is one edge connecting a vertex in X^{Δ} and a vertex in Y^{δ}. So we can always assume that there is at least one vertex $x^*\in X^{\Delta}$ and one vertex $y^*\in Y^{\delta}$ such that $x^*y^*\in E(G)$.

Let $x_1=x^*, x_2,\ldots,x_\delta$ be the neighbors of y^*. Choose δ vertices $y_1,y_2,\ldots,y_\delta$ in $Y\setminus\{y^*\}$. Then we have $d(x_1)\ge d(y_1)+2$ and $d(x_i)\ge d(y_i)+1$ for $i=2,3,\ldots,\delta$ by Claim 2. Let G^* be the graph obtained from G by deleting the edges x_iy^* and adding the edges y^*y_i for $i=1,2,\ldots,\delta$. Clearly G^* is still a bipartite graph. Then we have

$$\begin{aligned}\sigma_2(G^*)-\sigma_2(G)&=\sum_{i=1}^{\delta}[(d(x_i)-1)^2+(d(y_i)+1)^2-d(x_i)^2-d(y_i)^2]\\&=2(d(y_1)-d(x_1)+1)+2\sum_{i=2}^{\delta}(d(y_i)-d(x_i)+1)\\&<0,\end{aligned}$$

a contradiction.

Suppose $\Delta-\delta=1$. Then it is easy to see that $\Delta=\Delta(X)=t+1$, $\delta=\delta(Y)=t$. It follows from $\sum_{x\in X} d(x)=\sum_{y\in Y} d(y)$ that

$$|X^{\Delta}|(t+1)+|X^{\Delta-1}|t=|Y^{\delta+1}|(t+1)+|Y^{\delta}|t.$$

Then

$$|X^{\Delta}|=(|Y^{\delta+1}|+|Y^{\delta}|-|X^{\Delta}|-|X^{\Delta-1}|)t+|Y^{\delta+1}|\ge t+|Y^{\delta+1}|. \tag{1}$$

So we have

$$|X^{\Delta}|+|Y^{\delta+1}|\ge t+2|Y^{\delta+1}|\ge t. \tag{2}$$

Since $|X|=|X^{\Delta}|+|X^{\Delta-1}|<|Y|=|Y^{\delta+1}|+|Y^{\delta}|$, by (1) we have

$$|Y^{\delta}|>|X^{\Delta}|+|X^{\Delta-1}|-|Y^{\delta+1}|\ge t+|X^{\Delta-1}|.$$

This implies that

$$|X^{\Delta}| + |Y^{\delta+1}| = n - |X^{\Delta-1}| - |Y^{\delta}| \le n - |Y^{\delta}| < n - (t + |X^{\Delta-1}|) \le n - t. \tag{3}$$

By $|X^{\Delta}| + |Y^{\delta+1}| = r = 2m - nt$, and (2) and (3), we have

$$nt + t \le 2m < nt + n - t,$$

contradicting our assumption that $nt \le 2m < nt + t$ or $nt + n - t + 1 \le 2m < nt + n$.

Therefore, we obtain $\Delta - \delta = 2$. $\square$

Claim 5. $\delta(G) = \begin{cases} t-1, & \text{if } nt \le 2m < nt + t; \\ t, & \text{if } nt + n - t + 1 \le 2m < nt + n. \end{cases}$

Proof. If $\delta \le t - 2$, then

$$\begin{aligned} 2m &= |X^{\Delta}|(\delta + 2) + |X^{\Delta-1}|(\delta + 1) + |Y^{\delta+1}|(\delta + 1) + |Y^{\delta}|\delta \\ &\le (|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\delta+1}| + |Y^{\delta}|)t - |X^{\Delta-1}| - |Y^{\delta+1}| - 2|Y^{\delta}| \\ &\le nt - 2, \end{aligned}$$

a contradiction. If $\delta \ge t + 1$, then

$$\begin{aligned} 2m &= |X^{\Delta}|(\delta + 2) + |X^{\Delta-1}|(\delta + 1) + |Y^{\delta+1}|(\delta + 1) + |Y^{\delta}|\delta \\ &\ge (|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\delta+1}| + |Y^{\delta}|)(t + 1) + 2|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\delta+1}| \\ &\ge n(t + 1) + 2, \end{aligned}$$

a contradiction. So we have $\delta = t$ or $t - 1$.

Suppose $nt \le 2m < nt + t$. If $\delta = t$, then

$$\begin{aligned} 2m &= |X^{\Delta}|(t + 2) + |X^{\Delta-1}|(t + 1) + |Y^{\delta+1}|(t + 1) + |Y^{\delta}|t \\ &= nt + 2|X^{\Delta}| + |X^{\Delta-1}| + |Y^{\Delta}| \\ &> nt + t, \end{aligned}$$

a contradiction. Therefore, we have $\delta = t - 1$ when $nt \le 2m < nt + t$.

Suppose $nt + n - t + 1 \le 2m < nt + n$. If $\delta = t - 1$, then from $|X^{\Delta}| \le |X| \le \frac{n-1}{2}$ and $|Y^{\delta}| \ge 1$ we have

$$\begin{aligned} 2m &= |X^{\Delta}|(t + 1) + |X^{\Delta-1}|t + |Y^{\delta+1}|t + |Y^{\delta}|(t - 1) \\ &= nt + |X^{\Delta}| - |Y^{\delta}| \\ &\le nt + \frac{n-3}{2}. \end{aligned}$$

At the same time, noting that $nt + r = 2m \le \frac{n^2-1}{2}$, we obtain $t < \frac{n+1}{2}$. This implies that

$$2m \le nt + \frac{n-3}{2} < nt + n - t + 1,$$

a contradiction. Therefore, we have $\delta = t$ when $nt + n - t + 1 \le 2m < nt + n$. $\square$

Claim 6. $|X| = |Y| - 1$.

Proof. We prove this claim by contradiction. First, since $|X| + |Y| = n$ is odd, it is clear that $|X| \ne |Y| - 2$.

Now suppose $|X| \le |Y| - 3$. As in the proof of Claim 4, we can assume that there exists an edge $x^*y^* \in E(G)$ with $x^* \in X^{\Delta}$ and $y^* \in Y^{\delta}$. If $|Y^{\delta}| \ge \delta + 1$, then denote the neighbors of y^* by $x_1 = x^*, x_2, x_3, \ldots, x_{\delta}$ and choose δ vertices $y_1, y_2, \ldots, y_{\delta}$ in $Y^{\delta} \setminus \{y^*\}$. Let G' be the graph obtained from G by deleting the edges $x_i y^*$ and adding the edges $y^* y_i$ for $i = 1, 2, \ldots, \delta$. Clearly G' is still a bipartite graph. It is easy to see that $\sigma_2(G') - \sigma_2(G) < 0$, a contradiction. So we have $|Y^{\delta}| \le \delta$.

It follows from $|X^{\Delta}|(\delta+2)+|X^{\Delta-1}|(\delta+1)=|Y^{\delta+1}|(\delta+1)+|Y^{\delta}|\delta$ that

$$\begin{aligned}|X^{\Delta}| &= (|Y^{\delta+1}|+|Y^{\delta}|-|X^{\Delta}|-|X^{\Delta-1}|)\delta+|Y^{\delta+1}|-|X^{\Delta}|-|X^{\Delta-1}|\\ &\geq 3\delta+|Y^{\delta+1}|-|X^{\Delta}|-|X^{\Delta-1}|\\ &\geq 2\delta+|Y^{\delta+1}|+|Y^{\delta}|-|X^{\Delta}|-|X^{\Delta-1}|\\ &\geq 2\delta+3.\end{aligned}$$

Now let B denote the graph $B^s(n,m)$. From Claim 5 we know that $\delta(B)=\delta(G)=\delta$. Then we have

$$\begin{aligned}\sigma_2(B) &= n_{\Delta(B)}(\delta+2)^2+n_{\delta(B)}\delta^2+(n-n_{\Delta(B)}-n_{\delta(B)})(\delta+1)^2\\ &< n_{\Delta(B)}(\delta+2)^2+(n-n_{\Delta(B)})(\delta+1)^2\end{aligned}$$

and

$$\begin{aligned}\sigma_2(G) &= |X^{\Delta}|(\delta+2)^2+|Y^{\delta}|\delta^2+(n-|X^{\Delta}|-|Y^{\delta}|)(\delta+1)^2\\ &= (|X^{\Delta}|-|Y^{\delta}|)(\delta+2)^2+(n-|X^{\Delta}|+|Y^{\delta}|)(\delta+1)^2+|Y^{\delta}|((\delta+2)^2+\delta^2-2(\delta+1)^2)\\ &> (|X^{\Delta}|-|Y^{\delta}|)(\delta+2)^2+(n-|X^{\Delta}|+|Y^{\delta}|)(\delta+1)^2.\end{aligned}$$

At the same time, from Table 1, we can see that $n_{\Delta(B)}\leq\delta$. It follows from $|X^{\Delta}|\geq 2\delta+3$ and $|Y^{\delta}|\leq\delta$ that

$$\begin{aligned}\sigma_2(B)-\sigma_2(G) &< (n_{\Delta(B)}+|Y^{\delta}|-|X^{\Delta}|)((\delta+2)^2-(\delta+1)^2)\\ &\leq -3((\delta+2)^2-(\delta+1)^2)\\ &< 0,\end{aligned}$$

a contradiction. □

Claim 7. *$|X^{\Delta}|=m-\frac{(n-1)(\delta+1)}{2}$, $|X^{\Delta-1}|=\frac{(n-1)(\delta+2)}{2}-m$, $|Y^{\delta+1}|=m-\frac{(n+1)\delta}{2}$ and $|Y^{\delta}|=\frac{(n+1)(\delta+1)}{2}-m$.*

Proof. By Claim 6 we have

$$|X^{\Delta}|+|X^{\Delta-1}|=\frac{n-1}{2} \tag{4}$$

and

$$|Y^{\delta+1}|+|Y^{\delta}|=\frac{n+1}{2}. \tag{5}$$

On the other hand,

$$|X^{\Delta}|(\delta+2)+|X^{\Delta-1}|(\delta+1)=|Y^{\delta+1}|(\delta+1)+|Y^{\delta}|\delta=m. \tag{6}$$

Solving Eqs. (4)–(6), we get $|X^{\Delta}|=m-\frac{(n-1)(\delta+1)}{2}$, $|X^{\Delta-1}|=\frac{(n-1)(\delta+2)}{2}-m$, $|Y^{\delta+1}|=m-\frac{(n+1)\delta}{2}$ and $|Y^{\delta}|=\frac{(n+1)(\delta+1)}{2}-m$. □

From Claims 4, 5 and 7, we know that $\Delta(G)-\delta(G)=2$, $n_{\delta(G)}=\frac{(n+1)(\delta+1)}{2}-m$, $n_{\delta(G)+1}=n-\delta-1$ and $n_{\delta(G)+2}=m-\frac{(n-1)(\delta+1)}{2}$, where

$$\delta=\begin{cases}t-1, & \text{if } nt\leq 2m<nt+t;\\ t, & \text{if } nt+n-t+1\leq 2m<nt+n.\end{cases}$$

So the graph $B^s(n,m)$ has the same degree sequence as G. Therefore, $\sigma_2(B^s(n,m))$ attains the minimum value among all the bipartite graph with n vertices and m edges. This completes the proof of the theorem. □

Corollary 1. *Let n and m be two integers with $n\geq 2$ and $0\leq m\leq\lfloor n/2\rfloor\lceil n/2\rceil$, and G a bipartite graph with n vertices and m edges. Then the minimum possible value of $\sigma_2(G)$ is*

$$\begin{cases}(4m-n-nt)t+2m, & \text{if } n \text{ is even; or } n \text{ is odd and } nt+t\leq 2m\leq nt+n-t-1;\\ (4m+1-nt)t, & \text{if } n \text{ is odd and } nt\leq 2m<nt+t;\\ (4m-n+1-nt)(t+1), & \text{if } n \text{ is odd and } nt+n-t+1\leq 2m<nt+n,\end{cases}$$

where $t=\lfloor\frac{2m}{n}\rfloor$.

4. Maximum value of the sum of squares of degrees

Theorem 2. *Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$. Then $\sigma_2(B^l(n,m))$ attains the maximum value among all the bipartite graphs with n vertices and m edges.*

Proof. Suppose that G is a bipartite graph such that $\sigma_2(G)$ attains the maximum value among all the bipartite graphs with n vertices and m edges. Let (X, Y) be the bipartition of G. Without loss of generality, we assume $|X| = k \geq \lceil \frac{n}{2} \rceil$. Let $m = qk + r$, where $0 \leq r < k$.

First, let us prove that $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor \leq 1$. Suppose $m = \lfloor \frac{m}{k+1} \rfloor (k+1) + r'$, where $0 \leq r' < k+1$. If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor > 1$, then

$$\begin{aligned} r' &= \left\lfloor \frac{m}{k} \right\rfloor k + r - \left\lfloor \frac{m}{k+1} \right\rfloor (k+1) \\ &\geq \left\lfloor \frac{m}{k} \right\rfloor k + r - \left(\left\lfloor \frac{m}{k} \right\rfloor - 2 \right) (k+1) \\ &= r + 2(k+1) - \left\lfloor \frac{m}{k} \right\rfloor \\ &\geq r + 2(k+1) - \left\lfloor \frac{n}{2} \right\rfloor \\ &\geq r + 2(k+1) - k \\ &> k+1, \end{aligned}$$

a contradiction.

By Lemma 2, we can assume that q vertices in Y are all adjacent to all the vertices in X and one more vertex in Y is adjacent to r vertices in X. So we have

$$\begin{aligned} \sigma_2(G) &= r(q+1)^2 + (k-r)q^2 + qk^2 + r^2 \\ &= (m-qk)(q+1)^2 + (k+qk-m)q^2 + qk^2 + (m-qk)^2 \\ &= q(k-1)(k+qk-2m) + m^2 + m \\ &= \left\lfloor \frac{m}{k} \right\rfloor (k-1) \left(k + \left\lfloor \frac{m}{k} \right\rfloor k - 2m \right) + m^2 + m. \end{aligned}$$

Set $f(k) = \sigma_2(G)$. Then

$$f(k+1) - f(k) = \left\lfloor \frac{m}{k+1} \right\rfloor k \left(k + 1 + \left\lfloor \frac{m}{k+1} \right\rfloor (k+1) - 2m \right) - \left\lfloor \frac{m}{k} \right\rfloor (k-1) \left(k + \left\lfloor \frac{m}{k} \right\rfloor k - 2m \right).$$

If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor = 0$, then

$$f(k+1) - f(k) = 2 \left\lfloor \frac{m}{k} \right\rfloor \left(k \left(\left\lfloor \frac{m}{k} \right\rfloor + 1 \right) - m \right) \geq 0.$$

If $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{m}{k+1} \rfloor = 1$, then

$$f(k+1) - f(k) = 2 \left(\left\lfloor \frac{m}{k} \right\rfloor - k \right) \left(\left\lfloor \frac{m}{k} \right\rfloor k - m \right) \geq 0.$$

Thus, $f(k)$ is a nondecreasing function. So we can assume that $k = k_0 = \max\{k | m = qk + r, 0 \leq r < k, \lceil n/2 \rceil \leq k \leq n - q - \mathrm{sgn}(r)\}$. The proof follows from the construction of $B^l(n,m)$ immediately. □

Corollary 2. *Let n and m be two integers with $n \geq 2$ and $0 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, and G a bipartite graph with n vertices and m edges. Then the maximum possible value of $\sigma_2(G)$ is*

$$\left\lfloor \frac{m}{k_0} \right\rfloor (k_0 - 1) \left(k_0 + \left\lfloor \frac{m}{k_0} \right\rfloor k_0 - 2m \right) + m^2 + m,$$

where $k_0 = \{k | m = qk + r, 0 \leq r < k, \lceil \frac{n}{2} \rceil \leq k \leq n - q - \mathrm{sgn}(r)\}$.

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