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# Spreading fronts in a partially degenerate integro-differential reaction–diffusion system

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**Abstract.** This paper is concerned with the spreading and vanishing of an epidemic disease, which is described by a partially degenerate reaction–diffusion system with the nonlocal term and double free boundaries. We first consider the sign of the corresponding principal eigenvalue, which is determined by some given conditions. Then, we get the sufficient conditions that ensure the disease spreading or vanishing. At last, when spreading occurs, some rough estimates of the asymptotic spreading speed are given under some conditions.

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**Keywords.** Epidemic model, Partially degenerate, Nonlocal, Free boundary, Spreading and vanishing.

## 1. Introduction

The spreading phenomenon of the epidemic is one of the important subjects in mathematical epidemiology. To model the spatial spreading of a class of bacteria or viral diseases, Capasso and Maddalena [7] proposed the following reaction–diffusion model

$$\begin{cases} u_t = d\Delta u - au + cv, & (t,x) \in (0,+\infty)\times\Omega, \\ v_t = -bv + G(u), & (t,x) \in (0,+\infty)\times\Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}} + \alpha u = 0,\ \frac{\partial v}{\partial \boldsymbol{n}} + \alpha v = 0, & (t,x) \in (0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u(t,x)$ and $v(t,x)$ represent the spatial densities of the infectious agents and the infective human population at time $t$ and location $x$, respectively. The positive constant $d$ stands for the diffusion rate of the agents. Particularly, for malaria, the mosquito is the main spreading agents. For this case, the diffusion coefficient for the infectious agents will be much larger than that for the infective human population. Therefore, the diffusion coefficient of the infective human population can be set to zero. The coefficients $a,\ b$ and $c$ are all positive constants, and $\frac{1}{a}$ accounts for the mean lifetime of the infectious agents in the environment, $\frac{1}{b}$ measures the mean infectious period of the infective human population, and $c$ is the multiplicative factor of the infectious agents due to the human population. $G(u)$ denotes the force of infection on human population due to a concentration $u$ of the infectious agents.

For problem (1.1), Capasso and Maddalena [7] introduced a threshold parameter $\theta$ such that the epidemic eventually tends to extinction if $0 < \theta < 1$ or persistence if $\theta > 1$, where $\theta$ does not depend on the initial densities of the infectious agents and the infective human population. From the point of epidemic waves, the existence of Fisher type monotone traveling waves and minimal wave speed of problem (1.1) were obtained [40]. Wu [36] considered the existence of entire solutions of (1.1) in the bistable case.

To let the description of such a gradual spreading process be more close to the reality, the free boundary condition has been considered in more and more ecological models recently. For example, the readers can refer to [8–10,19,29,30,43] for single-species models. More works related to the system can

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be found, such as [11,14,15,25,27,32,35,42] for Lotka–Volterra competition systems, [28,31,34,39] for predator–prey systems, [18,21] for cooperative systems and [5,20] for epidemic models.

Recently, Ahn et al. [1] introduced the free boundary to describe the expanding fronts of an infective environment in problem (1.1), in which sufficient conditions for the bacteria to vanish or spread are given. Their main results reveal that if the multiplicative factor of the infectious agents is small, the epidemic will vanish eventually and the human population is safe. Otherwise, the spreading or vanishing of the epidemic depends on the initial infected habitat, the diffusion rate and the initial density. For this kind of partially degenerate reaction–diffusion systems, besides [1], Tarboush et al. [22] considered the spreading and vanishing in a West Nile virus model with expanding fronts, and Wang et al. [26] studied the spreading frontiers in the model which the reaction terms are described by more general form. Recently, Tian et al. [23] proposed a partially degenerate reaction–diffusion–advection model with free boundary to investigate the invasive process of Aedes aegypti mosquitoes.

However, some infectious agents $u$ at a point $x$ and time $t$ usually depend not only on the infective humans $v$ at the point $x$, but also on $v$ in a neighborhood of $x$, and even on $v$ in the whole region $\Omega$. To describe the mechanism of the infectious agents due to the infective human population better, Capasso [6] used $\int_\Omega K(x,y)v(t,y)\mathrm{d}y$ instead of $cv$ to model the growth rate of the agents. The reason why this is a global term is that the infective human population are moving, and then, the growth of the agents is related to the infective human population in a neighborhood of the original position. Hence, the growth of the agents can be represented as a spatial weighted average. For the above reasons, Capasso proposed an epidemic reaction–diffusion model described by the following system

$$\begin{cases} u_t = d\Delta u - au + \int_\Omega K(x,y)v(t,y)\mathrm{d}y, & (t,x) \in (0,+\infty)\times\Omega, \\ v_t = -bv + G(u), & (t,x) \in (0,+\infty)\times\Omega, \\ \beta(x)\frac{\partial u}{\partial \boldsymbol{n}} + \alpha(x)u = 0,\ \beta(x)\frac{\partial v}{\partial \boldsymbol{n}} + \alpha(x)v = 0, & (t,x) \in (0,+\infty)\times\partial\Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where the readers can refer to [4, Section 1.5.4] for what the functions $\beta(x)$ and $\alpha(x)$ mean. Capasso [6] introduced

$$\theta_1 = \frac{G'_+(0)}{(a+d\lambda^*)b}\frac{\max_{x\in\overline{\Omega}}\int_\Omega K(x,y)\phi_1(y)\mathrm{d}y}{\min_{x\in\overline{\Omega}}\phi_1(x)}$$

and

$$\theta_2 = \frac{G'_+(0)}{(a+d\lambda^*)b}\frac{\min_{x\in\overline{\Omega}}\int_\Omega K(x,y)\phi_1(y)\mathrm{d}y}{\max_{x\in\overline{\Omega}}\phi_1(x)},$$

where $\phi_1 \in H^1(\Omega)$ is the eigenfunction associated with the first eigenvalue $\lambda^*$ of $-\Delta$ in $\Omega$ with the condition $\beta(x)\frac{\partial \phi_1}{\partial \boldsymbol{n}} + \alpha(x)\phi_1 = 0$ on $\partial\Omega$. The author proved that if $\theta_1 < 1$, then the trivial solution is globally asymptotically stable in $X_+ = \{(u,v) \in X | (u,v) \geq 0\}$, where $X$ is an ordered Banach space with pointwise partial order. While for $\theta_2 > 1$, the system admits a unique nontrivial equilibrium which is globally asymptotically stable in $X_+\backslash\{(0,0)\}$. For problem (1.2), Xu and Zhao [37] obtained the asymptotic speed of spread for solutions and minimal wave speed of monotone traveling waves.

Motivated by the work of Ahn et al. [1], we will introduce the free boundary condition to describe such a gradual spreading process of the epidemic described by problem (1.2). The main purpose of this paper is to study the effect of the nonlocal term on the spreading of epidemic disease. Meanwhile, we want to see whether the modified model can explain the reality better than problem (1.2). For these reasons, we think it is worth studying.

We assume that the infectious agents and the infective population occupy a common region $[-h_0, h_0]$ with population density $u_0(x)$ and $v_0(x)$ at the very early stage. The varying infected environment is denoted by $[g(t), h(t)]$. Since the spread of epidemic discussed here is mainly due to the growth of infectious agents which results from the infective human population, it is reasonable to assume that the free boundary is caused only by the infectious agents. Just like the Stefan condition in [3], we can further assume that the spreading front expands at a speed that is proportional to the gradient of the infectious agents' population density at the front, namely $h'(t) = -\mu_1 u_x(t, h(t))$ and $g'(t) = -\mu_2 u_x(t, g(t))$. For simplicity, we mainly focus on the case that $\mu_1 = \mu_2 = \mu$. For the case that $\mu_1 \neq \mu_2$, we leave it for further consideration. For a nonlinear diffusion equation with different moving parameters, the readers can refer to [41].

In consideration of the above reasons, we shall use the following free boundary problem to describe the spreading of the epidemic disease,

$$\begin{cases} u_t = \mathrm{d}u_{xx} - au + \int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y, & t>0,\ g(t)<x<h(t),\\ v_t = -bv + G(u), & t>0,\ g(t)<x<h(t),\\ u(t,x) = v(t,x) = 0, & t\geq 0,\ x\leq g(t) \text{ or } x \geq h(t),\\ g(0) = -h_0,\ g'(t) = -\mu u_x(t,g(t)), & t>0,\\ h(0) = h_0,\ h'(t) = -\mu u_x(t,h(t)), & t>0,\\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & -h_0<x<h_0. \end{cases} \tag{1.3}$$

We are interested in the dynamics of positive solution $(u(t,x), v(t,x), g(t), h(t))$ of above problem. In (1.3), $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined with $u(t,x)$ and $v(t,x)$. The constants $d, a, b, h_0$ and $\mu$ are positive. Define $I_0 \doteq (-h_0, h_0)$,

$$\mathscr{X}_1(h_0) \doteq \left\{u_0(x) \in W_p^2(I_0)\colon u_0(x) > 0 \text{ for } x \in I_0,\ u_0(x) = 0 \text{ for } x \in \mathbb{R}\backslash I_0\right\},$$
$$\mathscr{X}_2(h_0) \doteq \left\{v_0(x) \in C(I_0)\colon v_0(x) > 0 \text{ for } x \in I_0,\ v_0(x) = 0 \text{ for } x \in \mathbb{R}\backslash I_0\right\},$$

where $p > 3$. The initial function $(u_0(x), v_0(x)) \in \mathscr{X}_1(h_0) \times \mathscr{X}_2(h_0)$. Assume that $K(x)$ satisfies

(**K**) $K \in C^1(\mathbb{R})$ is nonnegative, symmetric and $\int_{\mathbb{R}} K(x-y)\mathrm{d}y = 1$ for any $x \in \mathbb{R}$.

and there exists $M^*$ such that $G(u)$ satisfies the following two conditions:

(**A1**) $G \in C^2([0,\infty)),\ G(0) = 0,\ 0 < G'(z) \leq M^*,\ \forall z \geq 0;$

(**A2**) $\frac{G(z)}{z}$ is decreasing and

$$\limsup_{z\to+\infty} \frac{G(z)}{z} < ab.$$

An example is $G(z) = \frac{z}{1+z}$. Here we emphasis that conditions (A1) and (A2) make sure that problem (1.3) has a positive constant equilibrium point. Otherwise, the solution may blow up. For this case, we leave it for further consideration.

The rest of this paper is organized as follows. The global existence, uniqueness and estimates of solution and comparison principle are given in Sect. 2. To establish the criteria for spreading and vanishing, in Sect. 3 we provide some basic results about the principal eigenvalue. Section 4 is devoted to the long time behavior of $(u, v)$, and we get a spreading–vanishing dichotomy and give the criteria for spreading and vanishing. In Sect. 5, we give the rough estimation of asymptotic spreading speed under some specific situation. A brief discussion will be presented in Sect. 6.

## 2. Preliminaries

Before studying problem (1.3), we should obtain the global existence and uniqueness of solutions to problem (1.3) at first. In fact, the proof is essentially the same as in [1]. Here, we give the main results as follows:

**Theorem 2.1.** *For any given* $(u_0(x), v_0(x)) \in \mathscr{X}_1(h_0) \times \mathscr{X}_2(h_0)$ *and any* $\alpha \in (0,1)$*, there is a* $T > 0$ *such that problem* (1.3) *admits a unique solution*

$$(u, v, g, h) \in W_p^{1,2}(D_T) \times C(D_T) \times \left[C^{1+\frac{\alpha}{2}}([0,T])\right]^2;$$

*moreover,*

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}(D_T)} + \|v\|_{C(D_T)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C,$$

*where*

$$D_T := \left\{(t,x) \in \mathbb{R}^2 : \ t \in [0,T], \ x \in [g(t), h(t)]\right\},$$

$C$ *and* $T$ *depend only on* $h_0$*,* $\alpha$*,* $\|u_0\|_{W_p^2([-h_0,h_0])}$ *and* $\|v_0\|_{C([-h_0,h_0])}$*.*

*Proof.* Since $v$ has no diffusion term, we can use $u$ to represent $v$, define

$$\mathscr{G}_T = \{g \in C^1([0,T]) : \ g(0) = -h_0, \ g'(t) \le 0, \ 0 \le t \le T\},$$
$$\mathscr{H}_T = \{h \in C^1([0,T]) : \ h(0) = h_0, \ h'(t) \ge 0, \ 0 \le t \le T\}.$$

We introduce the mapping

$$E_t(u)(t,x) = \begin{cases} u(t,x), & g(t) \le x \le h(t), \\ 0, & x < g(t) \text{ or } x > h(t); \end{cases}$$

then, for $0 \le t \le T$ and $x \in \mathbb{R}$,

$$v(t,x) = \mathrm{e}^{-bt}[E_0(v_0)(x) + \int_0^t \mathrm{e}^{bs} G(E_s(u))(s,x)\mathrm{d}s] \doteq H(t,x,u).$$

Then, $u$ satisfies

$$\begin{cases} u_t = \mathrm{d}u_{xx} - au + \int_{\mathbb{R}} K(x-y)H(t,y,u(t,y))\mathrm{d}y, & 0 < t \le T, \ g(t) < x < h(t), \\ u(t,x) = 0, & 0 < t \le T, \ x \le g(t) \text{ or } x \ge h(t), \\ g(0) = -h_0, \ g'(t) = -\mu u_x(t, g(t)), & 0 < t \le T, \\ h(0) = h_0, \ h'(t) = -\mu u_x(t, h(t)), & 0 < t \le T, \\ u(0,x) = u_0(x), & -h_0 < x < h_0. \end{cases} \tag{2.1}$$

For the existence and uniqueness of the solution $u$ in problem (2.1), we can prove it as those in [10]. We first straighten the free boundary by the transformation:

$$r = \frac{2x}{h(t)-g(t)} - \frac{h(t)+g(t)}{h(t)-g(t)},$$
$$u(t,x) = u(t, \frac{h(t)-g(t)}{2} r + \frac{h(t)+g(t)}{2}) = w(t,r).$$

Then, problem (2.1) can be transformed into

$$\begin{cases} w_t - A(t)w_{rr} - B(t,r)w_r = -aw + f(t,r,w), & 0 < t \le T, \ -1 < r < 1, \\ w(t,r) = 0, & 0 \le t \le T, \ r \le -1 \text{ or } r \ge 1, \\ w(0,r) = u_0(h_0 r), & -1 \le r \le 1, \end{cases} \tag{2.2}$$

and

$$\begin{cases} h'(t) = -\mu \frac{2}{h(t)-g(t)} w_r(t,1), \ g'(t) = -\mu \frac{2}{h(t)-g(t)} w_r(t,-1), & 0 \le t \le T, \\ h(0) = h_0, \ g(0) = -h_0, \end{cases} \tag{2.3}$$

where

$$A(t)=\frac{4d}{[h(t)-g(t)]^2},\quad B(t,r)=\frac{h'(t)-g'(t)}{h(t)-g(t)}r+\frac{h'(t)+g'(t)}{h(t)-g(t)}$$

and

$$f(t,r,w)=\int_{\mathbb{R}} K\left(\frac{h(t)-g(t)}{2}r+\frac{h(t)+g(t)}{2}-y\right) H\left(t,\frac{h(t)-g(t)}{2}y+\frac{h(t)+g(t)}{2},w(t,y)\right)\mathrm{d}y.$$

Let $g^*=-\frac{\mu}{h_0}u_0'(-h_0)$ and $h^*=-\frac{\mu}{h_0}u_0'(h_0)$. For $0<T\le\frac{h_0}{2(1+|g^*|+h^*)}$, denote $I_T=[0,T]\times[-1,1]$,

$$D_T^1=\{w\in C(I_T):\ w(t,\pm1)=0,\ w(0,r)=u_0(h_0r),\ \|w-u_0\|_{C(I_T)}\le 1\},$$
$$D_T^2=\{g\in C^1([0,T]):\ g(0)=-h_0,\ g'(0)=g^*,\ \|g'-g^*\|_\infty\le 1\},$$
$$D_T^3=\{h\in C^1([0,T]):\ h(0)=h_0,\ h'(0)=h^*,\ \|h'-h^*\|_\infty\le 1\}.$$

Clearly, $D_T=D_T^1\times D_T^2\times D_T^3$ is a bounded and closed convex set of $C(I_T)\times C^1([0,T])\times C^1([0,T])$. When $g\in D_T^2$ and $h\in D_T^3$, we have

$$|g(t)+h_0|\le T\|g'\|_\infty\le T(1+g^*)<\frac{h_0}{2},$$
$$|h(t)-h_0|\le T\|h'\|_\infty\le T(1+h^*)<\frac{h_0}{2}.$$

Since $K\in C^1(R)$, $v_0(x)\in\mathscr{X}_2(h_0)$ and $G\in C^2([0,\infty))$, we can compute that

$$\begin{aligned}\|f(t,r,w)\|_{L^p(I_T)}^p&\le\int_{-1}^{1}\int_0^T\left[\int_{-1}^{1}K\left(\frac{h(t)-g(t)}{2}r+\frac{h(t)+g(t)}{2}-y\right)\right.\\&\qquad\left.H\left(t,\frac{h(t)-g(t)}{2}y+\frac{h(t)+g(t)}{2},w(t,y)\right)\mathrm{d}y\right]^p\mathrm{d}t\mathrm{d}r\\&\le 2C_1^p\int_0^T\left[\int_{-1}^{1}\mathrm{e}^{-bt}\left(v_0(h_0y)+\int_0^t\mathrm{e}^{bs}G(w)(s,y)\mathrm{d}s\right)\mathrm{d}y\right]^p\mathrm{d}t\\&\le 2C_1^p\int_0^T\left\{2\left[\left(C_2-\frac{C_3}{b}\right)\mathrm{e}^{-bt}+\frac{C_3}{b}\right]\right\}^p\mathrm{d}t<\infty,\end{aligned}$$

where

$$C_1=\max_{(t,r,y)\in I_T\times[-1,1]}K\left(\frac{h(t)-g(t)}{2}r+\frac{h(t)+g(t)}{2}-y\right),$$

and

$$C_2=\max_{y\in[-1,1]}v_0(h_0y),\quad C_3=\max_{(s,y)\in[0,T]\times[-1,1]}G(w)(s,y).$$

Applying standard $L^p$ theory and Sobolev imbedding theorem, we find, for any given $(w,g,h)\in D_T$, problem

$$\begin{cases}\widetilde{w}_t-A(t)\widetilde{w}_{rr}-B(t,r)\widetilde{w}_r=-aw+f(t,r,w), & 0<t\le T,\ -1<r<1,\\ \widetilde{w}(t,r)=0, & 0\le t\le T,\ r\le-1\text{ or }r\ge1,\\ \widetilde{w}(0,r)=u_0(h_0r), & -1\le r\le 1\end{cases}\tag{2.4}$$

has a unique solution $\widetilde{w}(t,r) \in W_p^{1,2}(I_T) \hookrightarrow C^{\frac{1+\alpha}{2},1+\alpha}(I_T)$, and

$$\|\widetilde{w}\|_{W_p^{1,2}(I_T)} + \|\widetilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(I_T)} \le C_4.$$

By the continuous dependence on the given data, $\widetilde{w}$ in $C^{\frac{1+\alpha}{2},1+\alpha}(I_T)$ depends continuously on $(w,g,h) \in D_T$. For such $\widetilde{w}$, problem (2.3) has a unique solution $(\widetilde{g},\widetilde{h})$,

$$\widetilde{g}(0) = -h_0,\ \widetilde{h}(0) = h_0,\ \widetilde{g}'(0) = g^*,\ \widetilde{h}'(0) = h^*$$

and

$$(\widetilde{g}',\widetilde{h}') \in [C^{\frac{\alpha}{2}}([0,T])]^2,\ \|g'\|_{C^{\frac{\alpha}{2}}([0,T])} + \|h'\|_{C^{\frac{\alpha}{2}}([0,T])} \le C_5. \tag{2.5}$$

Now we define a mapping $\mathscr{F}:\ D_T \to [C^1([0,T])]^2$ by

$$\mathscr{F}(w,g,h) = (\widetilde{w},\widetilde{g},\widetilde{h}).$$

By (2.5), we have

$$\begin{aligned}
&\|\widetilde{w} - u_0\|_{C(I_T)} \le \|\widetilde{w} - u_0\|_{C^{\frac{1+\alpha}{2},0}(I_T)} T^{\frac{1+\alpha}{2}} \le C_4 T^{\frac{1+\alpha}{2}},\\
&\|\widetilde{g}' - g^*\|_{C([0,T])} \le \|\widetilde{g}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le C_5 T^{\frac{\alpha}{2}},\\
&\|\widetilde{h}' - h^*\|_{C([0,T])} \le \|\widetilde{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le C_5 T^{\frac{\alpha}{2}},
\end{aligned}$$

Therefore, if $T \le \min\left\{\frac{h_0}{2(1+|g^*|+h^*)},\ C_4^{-\frac{2}{1+\alpha}},\ C_5^{-\frac{2}{\alpha}}\right\}$, then $\mathscr{F}$ maps $D_T$ into itself.

Next we prove that $\mathscr{F}$ is a contraction mapping on $D_T$ for $T > 0$ sufficiently small. Let $(w_i,g_i,h_i) \in D_T (i = 1,2)$, and denote $(\widetilde{w}_i,\widetilde{g}_i,\widetilde{h}_i) = \mathscr{F}(w_i,g_i,h_i)$. Set $\omega = \widetilde{w}_1 - \widetilde{w}_2$. We can see that $\omega$ satisfies

$$\begin{cases}
\omega_t - A_1(t)\omega_{rr} - B_1(t,r)\omega_r = -a\omega + (A_1(t) - A_2(t))\widetilde{w}_{2rr} & \\
\quad +(B_1(t,r) - B_2(t,r))\widetilde{w}_{2r} + f_1(t,r,w_1) - f_2(t,r,w_2), & 0 < t \le T, \quad -1 < r < 1,\\
\omega(t,r) = 0, & 0 < t \le T, \quad r \le -1 \text{ or } r \ge 1,\\
\omega(0,r) = 0, & -1 \le r \le 1,
\end{cases} \tag{2.6}$$

where

$$A_i(t) = \frac{4d}{[h_i(t) - g_i(t)]^2}, \quad B_i(t,r) = \frac{h_i'(t) - g_i'(t)}{h_i(t) - g_i(t)} r + \frac{h_i'(t) + g_i'(t)}{h_i(t) - g_i(t)}$$

and

$$\begin{aligned}
f_i(t,r,w_i) = \int_{\mathbb{R}} & K\left(\frac{h_i(t) - g_i(t)}{2} r + \frac{h_i(t) + g_i(t)}{2} - y\right)\\
& H\left(t, \frac{h_i(t) - g_i(t)}{2} y + \frac{h_i(t) + g_i(t)}{2}, w_i(t,y)\right) \mathrm{d}y.
\end{aligned}$$

The rest of proof can be done by those in [10]. By applying the $L^p$ estimates for parabolic equations and Sobolev's imbedding theorem, we can deduce that $\mathscr{F}$ is a contraction mapping on $D_T$ for $T > 0$ sufficiently small. Then, it follows from the contraction mapping theorem that $\mathscr{F}$ has a unique fixed point $(w,g,h)$ in $D_T$; namely, (2.2) and (2.3) have a unique solution $(w,g,h)$. Then, (2.1) has a unique solution $(u,g,h)$. $v$ can be derived by $(u,g,h)$. Hence, problem (1.3) exists a unique solution $(u,v,g,h)$. □

To get the global existence and uniqueness, we need the following comparison principle to derive the estimates of the solution to problem (1.3).

**Theorem 2.2.** (Comparison principle) *Assume that*

$$\overline{g},\ \overline{h} \in C^1([0,+\infty)),\quad \overline{u}(t,x) \in C(\overline{D}) \cap C^{1,2}(D),\quad \overline{v}(t,x) \in C(\overline{D}) \cap C^{1,0}(D)$$

*with*

$$D := \left\{(t,x) \in \mathbb{R}^2 :\ 0 < t < \infty,\ \overline{g}(t) < x < \overline{h}(t)\right\},$$

*and*

$$\begin{cases} \overline{u}_t \geq d\overline{u}_{xx} - a\overline{u} + \int_{\mathbb{R}} K(x-y)\overline{v}(t,y)\mathrm{d}y, & t > 0,\ \overline{g}(t) < x < \overline{h}(t),\\ \overline{v}_t \geq -b\overline{v} + G(\overline{u}), & t > 0,\ \overline{g}(t) < x < \overline{h}(t),\\ \overline{u}(t,x) = \overline{v}(t,x) = 0, & t \geq 0,\ x \leq \overline{g}(t)\ \text{or}\ x \geq \overline{h}(t),\\ \overline{g}(0) \leq -h_0,\ \overline{g}'(t) \leq -\mu\overline{u}_x(t,\overline{g}(t)), & t > 0,\\ \overline{h}(0) \geq h_0,\ \overline{h}'(t) \geq -\mu\overline{u}_x(t,\overline{h}(t)), & t > 0,\\ \overline{u}(0,x) \geq u_0(x),\ \overline{v}(0,x) \geq v_0(x), & -h_0 < x < h_0. \end{cases} \tag{2.7}$$

*Then, the solution $(u,v,g,h)$ of the free boundary problem (1.3) satisfies*

$$\begin{aligned} &h(t) \leq \overline{h}(t),\quad g(t) \geq \overline{g}(t)\quad \text{for}\quad t \geq 0,\\ &u(t,x) \leq \overline{u}(t,x),\quad v(t,x) \leq \overline{v}(t,x)\quad \text{for all } t \geq 0\quad \text{and}\quad g(t) \leq x \leq h(t). \end{aligned}$$

*Proof.* The proof can be done by using the argument of [35, Lemma 2.3]. □

**Remark 2.3.** The pair $(\overline{u},\overline{v},\overline{g},\overline{h})$ in Theorem 2.2 is usually called an upper solution of (1.3). We can define a lower solution $(\underline{u},\underline{v},\underline{g},\underline{h})$ by reversing all of the inequalities in appropriate places.

For $(\omega_1(x),\omega_2(x)) \in \mathscr{X}_1(h_0) \times \mathscr{X}_2(h_0)$, let $(u_0,v_0) = \sigma(\omega_1,\omega_2)$. We write $(u^\sigma,v^\sigma,g^\sigma,h^\sigma)$ to emphasize the dependence of the solution on $\sigma$. The following corollary results directly from Theorem 2.2.

**Corollary 2.4.** *If $\sigma_1 \leq \sigma_2$, then $u^{\sigma_1} \leq u^{\sigma_2}$ and $v^{\sigma_1} \leq v^{\sigma_2}$ for all $t \geq 0$ and $g^{\sigma_1} \leq x \leq h^{\sigma_1}$, $g^{\sigma_1} \geq g^{\sigma_2}$ and $h^{\sigma_1} \leq h^{\sigma_2}$ for all $t \geq 0$.*

Now we exhibit the estimates of the solution.

**Lemma 2.5.** *Let $(u,v,g,h)$ be a solution to (1.3) defined for $t \in (0,T_0]$, where $T_0 \in (0,+\infty)$. Then, there exist constants $M_1$, $M_2$ and $M_3$ independent of $T_0$ such that:*

$$\begin{aligned} &0 < u(t,x) \leq M_1\quad \text{for all } 0 < t \leq T_0\quad \text{and}\quad g(t) < x < h(t),\\ &0 < v(t,x) \leq M_2\quad \text{for all } 0 < t \leq T_0\quad \text{and}\quad g(t) < x < h(t),\\ &0 < -g'(t),\quad h'(t) \leq M_3\quad \text{for all } 0 < t \leq T_0. \end{aligned}$$

*Proof.* The positivity of $u$ and $v$ is obvious. Next we consider its upper bounds. Note that $\lim_{z\to\infty}\frac{G(z)}{z} < ab$ by condition (A2), there exist $M_1$ and $M_2$ such that

$$M_1 \geq u_0(x),\quad M_2 \geq v_0(x)\quad \text{in}\quad [-h_0,h_0],\quad -aM_1 + M_2 < 0,\quad -bM_2 + G(M_1) < 0.$$

Denote $D_{T_0} \doteq \{(t,x) \in \mathbb{R}^2 :\ 0 \leq t \leq T_0,\ g(t) \leq x \leq h(t)\}$. Let

$$(U(t,x),V(t,x)) = (M_1 - u, M_2 - v)\mathrm{e}^{-(1+M^*)t}$$

for $0 \leq t \leq T_0$ and $x \in \mathbb{R}$, then

$$G(u) = G(M_1 - U\mathrm{e}^{(1+M^*)t}) = G(M_1) - G'(\xi)U\mathrm{e}^{(1+M^*)t},$$

where $M^*$ is given in condition (A1) and $\xi$ is between $M_1$ and $u$. Meanwhile, $(U,V)$ satisfies:

$$\begin{cases} U_t > \mathrm{d}U_{xx} - (a+1+M^*)U & \\ \quad + \int_{\mathbb{R}} K(x-y)V(t,y)\mathrm{d}y, & 0 < t \le T_0,\ g(t) < x < h(t), \\ V_t = -(b+1+M^*)V + G(\xi)U, & 0 < t \le T_0,\ g(t) < x < h(t), \\ U(t,x) = M_1 \mathrm{e}^{-(1+M^*)t}, & 0 < t \le T_0,\ x \le g(t) \text{ or } x \ge h(t), \\ V(t,x) = M_2 \mathrm{e}^{-(1+M^*)t}, & 0 < t \le T_0,\ x \le g(t) \text{ or } x \ge h(t), \\ u(0,x) \ge 0,\ v(0,x) \ge 0, & -h_0 < x < h_0. \end{cases} \tag{2.8}$$

Next we claim that $\min\{U(t,x), V(t,x)\} \ge 0$ in $D_{T_0}$. Otherwise, there exists $(t_0, x_0) \in (0, T_0] \times (g(t), h(t))$ such that

$$\min\{U(t_0,x_0), V(t_0,x_0)\} = \min_{(t,x)\in D_{T_0}} \min\{U(t,x), V(t,x)\} < 0.$$

If $U(t_0,x_0) = \min\{U(t_0,x_0), V(t_0,x_0)\} < 0$, then $U_t(t_0,x_0) - dU_{xx}(t_0,x_0) \le 0$. However,

$$\begin{aligned} &-(a+1+M^*)U(t_0,x_0) + \int_{\mathbb{R}} K(x_0-y)V(t_0,y)\mathrm{d}y \\ &\quad > -(a+1+M^*)U(t_0,x_0) + \int_{g(t_0)}^{h(t_0)} K(x_0-y)V(t_0,y)\mathrm{d}y \\ &\quad > -(a+M^*)U(t_0,x_0) + \left( \int_{g(t_0)}^{h(t_0)} K(x_0-y)\mathrm{d}y - 1 \right) U(t_0,x_0) > 0, \end{aligned}$$

which leads a contradiction to the first inequality in (2.8). The case of that $V(t_0,x_0) = \min\{U(t_0,x_0), V(t_0,x_0)\} < 0$ can be done by the similar arguments of [1, Lemma 2.2]. Hence, $\min_{(t,x)\in D_{T_0}}(U(t,x), V(t,x)) \ge 0$, namely $u \le M_1$ and $v \le M_2$ in $D_{T_0}$.

The rest of the proof is similar to that of [10, Lemma 2.2], and we omit it here. □

The following conclusion about global existence and uniqueness of the solution results from Theorem 2.1 and Lemma 2.5.

**Theorem 2.6.** *The solution of* (1.3) *exists and is unique for all* $t > 0$.

It follows from Lemma 2.5 that $x = h(t)$ is strictly increasing and $x = g(t)$ is strictly decreasing. Hence, there exist $h_\infty,\ -g_\infty \in (0, +\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$ and $\lim_{t\to+\infty} g(t) = g_\infty$. The next lemma shows that $(g_\infty, h_\infty)$ can never be a half-infinite interval, and can be proved by the same arguments as in [1, Lemma 3.1] (see also [12, Lemma 2.8]).

**Lemma 2.7.** *Suppose that* $(u, v, g, h)$ *is a solution to* (1.3) *defined for all* $t \ge 0$ *and* $g(t) \le x \le h(t)$. *Then,*

$$-2h_0 < g(t) + h(t) < 2h_0 \text{ for } t \in [0, +\infty).$$

## 3. The principal eigenvalue

Now we study the principal eigenvalue of the following problem

$$\begin{cases} u_t = \mathrm{d}u_{xx} - au + \int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y, & t > 0,\ -l < x < l, \\ v_t = -bv + G(u), & t > 0,\ -l < x < l, \\ u(t,x) = v(t,x) = 0, & t > 0,\ x \le -l \text{ or } x \ge l. \end{cases} \tag{3.1}$$

Due to the nonlocal term, the method in [26] cannot be used directly. This needs us to make some modification.

Linearizing (3.1) at zero solution, we obtain

$$\begin{cases} u_t = \mathrm{d}u_{xx} - au + \int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y, & t>0,\ -l<x<l, \\ v_t = -bv + G'(0)u, & t>0,\ -l<x<l, \\ u(t,x) = v(t,x) = 0, & t>0,\ x \le -l \text{ or } x \ge l. \end{cases} \tag{3.2}$$

Define

$$\Pi(l) \doteq \left\{\phi \mid \phi \in H_0^1((-l,l)),\ \phi \ge 0 \text{ and } \phi \not\equiv 0 \text{ in } (-l,l),\ \phi = 0 \text{ in } \mathbb{R}\backslash(-l,l)\right\};$$

let $\phi, \psi \in \Pi(l)$. Substituting $u(t,x) = \mathrm{e}^{-\lambda t}\phi(x)$ and $v(t,x) = \mathrm{e}^{-\lambda t}\psi(x)$ into (3.2), we obtain the associated eigenvalue problem

$$\begin{cases} -\lambda\phi = d\phi'' - a\phi + \int_{\mathbb{R}} K(x-y)\psi(y)\mathrm{d}y, & -l<x<l, \\ -\lambda\psi = -b\psi + G'(0)\phi, & -l<x<l, \\ \phi(x) = \psi(x) = 0, & x \le -l \text{ or } x \ge l. \end{cases} \tag{3.3}$$

By a similar manner to [24, Lemma 3.4] (see also [38, Lemma 3.1]), we can prove the nonlocal elliptic eigenvalue problem (3.3) has a principal eigenvalue denoted by $\lambda_0$ which is associated with a strongly positive eigenvector $(\phi, \psi) \gg 0$. It is easy to see that $\phi = \theta\psi$, where

$$\theta = \frac{b-\lambda_0}{G'(0)}.$$

We have $\lambda_0 < b$. In fact, if $\lambda_0 \ge b$, then $\phi\psi = \frac{b-\lambda_0}{G'(0)}\psi^2 \le 0$, which is a contradiction.

For any $\phi \in \Pi(l)$, we substitute $\psi = \frac{G'(0)}{b-\lambda}\phi$ into the first equation of (3.3) to get the following eigenvalue problem

$$\begin{cases} -\lambda\phi = d\phi'' - a\phi + \frac{G'(0)}{b-\lambda}\int_{\mathbb{R}} K(x-y)\phi(y)\mathrm{d}y, & -l<x<l, \\ \phi(x) = 0, & x \le -l \text{ or } x \ge l. \end{cases} \tag{3.4}$$

Multiplying the first equation of (3.4) by $\phi$ and integrating over $(-l,l)$ give

$$-\lambda\int_{-l}^{l}\phi^2\mathrm{d}x = d\int_{-l}^{l}\phi''\cdot\phi\mathrm{d}x - a\int_{-l}^{l}\phi^2\mathrm{d}x + \frac{G'(0)}{b-\lambda}\int_{-l}^{l}\int_{-\infty}^{+\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x;$$

then,

$$-\lambda\int_{-l}^{l}\phi^2\mathrm{d}x = -d\int_{-l}^{l}\phi'^2\mathrm{d}x - a\int_{-l}^{l}\phi^2\mathrm{d}x + \frac{G'(0)}{b-\lambda}\int_{-l}^{l}\int_{-\infty}^{+\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x.$$

Then, simple computations yield

$$A\lambda^2 + B\lambda + C = 0, \tag{3.5}$$

where

$$A = \int_{-l}^{l}\phi^2\mathrm{d}x,\ B = -(a+b)\int_{-l}^{l}\phi^2\mathrm{d}x - d\int_{-l}^{l}\phi'^2\mathrm{d}x$$

and

$$C = ab\int_{-l}^{l} \phi^2 \mathrm{d}x + bd\int_{-l}^{l} \phi'^2 \mathrm{d}x - G'(0)\int_{-l}^{l}\int_{-\infty}^{+\infty} K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x.$$

Direct calculation yields

$$\begin{aligned} B^2 - 4AC &> \left[-(a+b)\int_{-l}^{l} \phi^2 \mathrm{d}x - d\int_{-l}^{l} \phi'^2 \mathrm{d}x\right]^2 - 4\int_{-l}^{l} \phi^2 \mathrm{d}x \left(ab\int_{-l}^{l} \phi^2 \mathrm{d}x + bd\int_{-l}^{l} \phi'^2 \mathrm{d}x\right) \\ &= \left[(a-b)\int_{-l}^{l} \phi^2 \mathrm{d}x + d\int_{-l}^{l} \phi'^2 \mathrm{d}x\right]^2 \geq 0, \end{aligned}$$

which implies (3.5) has two roots denoted by $\frac{-B\pm\sqrt{B^2-4AC}}{2A}$.

Next, we will prove the following lemma.

**Lemma 3.1.** *The lower bound of* $\frac{-B-\sqrt{B^2-4AC}}{2A}$ *exists for any* $\phi \in \Pi(l)$.

*Proof.* Since

$$\begin{aligned} &\int_{-l}^{l}\int_{-\infty}^{\infty} K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x \\ &\quad \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y)\frac{\phi^2(y)+\phi^2(x)}{2}\mathrm{d}y\mathrm{d}x \\ &\quad = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y)\mathrm{d}x\phi^2(y)\mathrm{d}y + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y)\mathrm{d}y\phi^2(x)\mathrm{d}x \\ &\quad = \int_{-l}^{l} \phi^2(x)\mathrm{d}x, \end{aligned}$$

we have

$$C \geq (ab - G'(0))\int_{-l}^{l} \phi^2(x)\mathrm{d}x + bd\int_{-l}^{l} \phi'^2(x)\mathrm{d}x.$$

Here, we should emphasis that the equality holds if and only if $K(x) = \delta(x)$. Define

$$s = \int_{-l}^{l} \phi^2(x)\mathrm{d}x \text{ and } t = \int_{-l}^{l} \phi'^2(x)\mathrm{d}x.$$

It is well known that $\frac{t}{s}(\phi)$ attains its minimum $\frac{\pi^2}{4l^2}$ at $\phi(x) = \cos\left(\frac{\pi}{2l}x\right)$. Direct calculation yields

$$\begin{aligned} \frac{-B-\sqrt{B^2-4AC}}{2A} &\geq \frac{(a+b)s + \mathrm{d}t - \sqrt{[(a+b)s+\mathrm{d}t]^2 - 4s[(ab-G'(0))s + b\mathrm{d}t]}}{2s} \\ &= \frac{a+b}{2} + \frac{d}{2}\cdot\frac{t}{s} - \sqrt{\left(\frac{a+b}{2} + \frac{d}{2}\cdot\frac{t}{s}\right)^2 - \left(ab - G'(0) + bd\frac{t}{s}\right)} := f_1\left(\frac{t}{s}\right). \end{aligned}$$

Let

$$r=\frac{a+b}{2}+\frac{d}{2}\cdot\frac{t}{s},$$

then

$$f_1\left(\frac{t}{s}\right)=f_1\left(\frac{2}{d}\left(r-\frac{a+b}{2}\right)\right)=r-\sqrt{r^2-2br+b^2+G'(0)}\doteq f_2(r).$$

It is easy to check that

$$f_2'(r)=1-\frac{r-b}{\sqrt{(r-b)^2+G'(0)}}>0.$$

In view of

$$r\geq\frac{a+b}{2}+\frac{d\pi^2}{8l^2},$$

then we have

$$f_2(r)\geq f_2\left(\frac{a+b}{2}+\frac{d\pi^2}{8l^2}\right).$$

Hence, the lower bound of $\frac{-B-\sqrt{B^2-4AC}}{2A}$ exists. □

By Lemma 3.1, we can define the infimum of $\frac{-B-\sqrt{B^2-4AC}}{2A}$. Then, the principal eigenvalue $\lambda_0$ of (3.3) has the following form:

$$\lambda_0=\inf_{\phi\in\Pi(l)}\left\{\frac{-B-\sqrt{B^2-4AC}}{2A}\right\}. \tag{3.6}$$

Consider the elliptic eigenvalue problem

$$\begin{cases}-d\Phi''-\int_{\mathbb{R}}K(x-y)\Psi(y)\mathrm{d}y+a\Phi=0, & -l<x<l,\\ b\Psi=\widehat{\lambda}G'(0)\Phi, & -l<x<l,\\ \Phi(x)=\Psi(x)=0, & -l\leq x \text{ or } x\geq l.\end{cases} \tag{3.7}$$

By arguments similar to those in [17], it follows that (3.7) has a principal eigenvalue denoted by $\widehat{\lambda}_0$ which is associated with a strongly positive eigenvector $(\Phi,\Psi)\gg 0$. $\widehat{\lambda}_0$ has the following variational formula

$$\widehat{\lambda}_0=\inf_{\Phi\in\Pi(l)}\left\{\frac{b\left(a\int_{-l}^{l}\Phi^2\mathrm{d}x+d\int_{-l}^{l}\Phi'^2\mathrm{d}x\right)}{G'(0)\int_{-l}^{l}\int_{-\infty}^{+\infty}K(x-y)\Phi(y)\mathrm{d}y\Phi(x)\mathrm{d}x}\right\}.$$

Define

$$\Theta(d,l)\doteq\frac{1}{\widehat{\lambda}_0}=\sup_{\Phi\in\Pi(l)}\left\{\frac{G'(0)\int_{-l}^{l}\int_{-\infty}^{+\infty}K(x-y)\Phi(y)\mathrm{d}y\Phi(x)\mathrm{d}x}{b\left(a\int_{-l}^{l}\Phi^2\mathrm{d}x+d\int_{-l}^{l}\Phi'^2\mathrm{d}x\right)}\right\};$$

we have the following lemma.

**Lemma 3.2.** *$1-\Theta(d,l)$ has the same sign as $\lambda_0$.*

*Proof.* When $\lambda_0\leq 0$, the proof can be done by modifying the argument of part (d) in [2, Lemma 2.3]. Here, we give the details below for completeness.

According to the definition of $\Theta(d,l)$, there exists a positive function $\Phi(x)\in\Pi(l)$ such that

$$-d\Phi''-\frac{G'(0)}{b\Theta(d,l)}\int_{-\infty}^{+\infty}K(x-y)\Phi(y)\mathrm{d}y+a\Phi(x)=0 \text{ in } (-l,l). \tag{3.8}$$

It follows from (3.4) and (3.8) that

$$d\phi'' - a\phi + \frac{G'(0)}{b-\lambda_0}\int_{-\infty}^{+\infty} K(x-y)\phi(y)\mathrm{d}y + \lambda_0\phi = 0,\ x\in(-l,l), \tag{3.9}$$

$$d\Phi'' + \frac{G'(0)}{b\Theta(d,l)}\int_{-\infty}^{+\infty} K(x-y)\Phi(y)\mathrm{d}y - a\Phi(x) = 0,\ x\in(-l,l). \tag{3.10}$$

Recall that $K(x)$ is symmetric and $\Phi=\phi=0$ in $\mathbb{R}\backslash(-l,l)$. We multiply (3.9) by $\Phi$ and (3.10) by $\phi$, integrate over $(-l,l)$ and subtract the resulting equations to obtain

$$G'(0)\int_{-l}^{l}\int_{-\infty}^{+\infty} K(x-y)\Phi(y)\mathrm{d}y\phi(x)\mathrm{d}x\left(\frac{1}{b\Theta(d,l)} - \frac{1}{b-\lambda_0}\right) = \lambda_0\int_{-l}^{l}\phi(x)\Phi(x)\mathrm{d}x.$$

Since

$$G'(0)\int_{-l}^{l}\int_{-\infty}^{+\infty} K(x-y)\Phi(y)\mathrm{d}y\phi(x)\mathrm{d}x$$

and $\int_{-l}^{l}\phi(x)\Phi(x)\mathrm{d}x$ are both positive, we conclude that $\frac{1}{b\Theta(d,l)} - \frac{1}{b-\lambda_0}$ and $\lambda_0$ have the same sign. Then, $1-\Theta(d,l)<0$ when $\lambda_0<0$, $1-\Theta(d,l)=0$ when $\lambda_0=0$.

While for $\lambda_0>0$, it follows directly from (3.5) and (3.6) that $C>0$. Then, from the definition of $\Theta(d,l)$, we know $1-\Theta(d,l)>0$. □

Next, we consider the elliptic eigenvalue problem

$$\begin{cases} -d\phi'' + a\phi - \frac{G'(0)}{b}\int_{-\infty}^{\infty} K(x-y)\phi(y)\mathrm{d}y = \gamma\phi, & -l<x<l, \\ \phi(x)=0, & x\le -l \text{ or } x\ge l. \end{cases} \tag{3.11}$$

We denote by $\gamma_0$ the unique principal eigenvalue of (3.11). It is well known that $\gamma_0$ has the following variational formula

$$\gamma_0 = \inf_{\phi\in\Pi(l)}\left\{\frac{d\int_{-l}^{l}\phi'^2\mathrm{d}x + a\int_{-l}^{l}\phi^2\mathrm{d}x - \frac{G'(0)}{b}\int_{-l}^{l}\int_{-\infty}^{\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x}\right\}.$$

Similarly to Lemma 3.2, we have the following lemma.

**Lemma 3.3.** $1-\Theta(d,l)$ *has the same sign as* $\gamma_0$.

It follows from Lemmas 3.2 and 3.3 that $\lambda_0$ and $\gamma_0$ have the same sign, which implies that we can give the sign of $\lambda_0$ by the conditions which determine the sign of $\gamma_0$. Next, we show the properties of $\gamma_0$.

In the following, we show the monotonicity and sign of $\gamma_0$ with respect to $l$ and $d$. We write $\gamma_0(l,d)=\gamma_0(l)$ for any fixed $d$ and varying $l$ for brevity. Similarly, we write $\gamma_0(l,d)=\gamma_0(d)$ for any fixed $l$ and varying $d$.

For any fixed $d$, we denote

$$P(l) = a - \frac{G'(0)}{b}\frac{\int_{-l}^{l}\int_{-l}^{l}K(x-y)\cos\left(\frac{\pi}{2l}y\right)\mathrm{d}y\cos\left(\frac{\pi}{2l}x\right)\mathrm{d}x}{\int_{-l}^{l}\cos^2\left(\frac{\pi}{2l}x\right)\mathrm{d}x}.$$

To study the sign of $\gamma_0$, we need the limit behavior of $P(l)$ as $l$ tends to $\infty$. Firstly, we give the following condition:

(H) $\lim_{l\to\infty}P(l)$ exists and $\lim_{l\to\infty}P(l)<0$.

**Lemma 3.4.** *For any fixed $d$, the following statements are valid:*

(i) $\gamma_0(l)$ *is strictly decreasing in* $l$.
(ii) $\lim_{l\to 0}\gamma_0(l) > 0$.
(iii) *If* $a - \frac{G'(0)}{b} > 0$, *then* $\gamma_0(l) > 0$ *for all* $l > 0$.
(iv) *If the condition (H) holds, then the equation* $\gamma_0(l) = 0$ *has a unique positive root denoted by* $l^*(d)$. *Furthermore, if* $0 < l < l^*(d)$, *then* $\gamma_0(l) > 0$, *and if* $l > l^*(d)$ *then* $\gamma_0(l) < 0$.

*Proof.* Part (i) can be done by the similar argument of part $(a)$ in [43, Theorem 3.2].

For part (ii), since

$$\frac{d\int_{-l}^{l}\phi'^2\mathrm{d}x + a\int_{-l}^{l}\phi^2\mathrm{d}x - \frac{G'(0)}{b}\int_{-l}^{l}\int_{-\infty}^{\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x} \geq \frac{d\pi^2}{4l^2} + a - \frac{G'(0)}{b},$$

we have $\gamma_0(l) \geq \frac{d\pi^2}{4l^2} + a - \frac{G'(0)}{b}$. Then, $\lim_{l\to 0}\gamma_0(l) > 0$.

For part (iii), since

$$\begin{aligned}
\gamma_0 &= \inf_{\phi\in\Pi(l)}\left\{\frac{d\int_{-l}^{l}\phi'^2\mathrm{d}x + a\int_{-l}^{l}\phi^2\mathrm{d}x - \frac{G'(0)}{b}\int_{-l}^{l}\int_{-\infty}^{\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x}\right\}\\
&\geq \inf_{\phi\in\Pi(l)}\left\{\frac{d\int_{-l}^{l}\phi'^2\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x}\right\} + a - \frac{G'(0)}{b}\sup_{\phi\in\Pi(l)}\left\{\frac{\int_{-l}^{l}\int_{-\infty}^{\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x}\right\}\\
&\geq \frac{d\pi^2}{4l^2} + a - \frac{G'(0)}{b},
\end{aligned}$$

We have $\lim_{l\to+\infty}\gamma_0(l) \geq a - \frac{G'(0)}{b} > 0$. Combining this with part (i), we can derive part (iii).

For part (iv), let

$$\chi(x) = \begin{cases}\cos\left(\frac{\pi}{2l}x\right), & x\in(-l,l),\\ 0, & x\in\mathbb{R}\backslash(-l,l);\end{cases}$$

since

$$\begin{aligned}
\gamma_0 &= \inf_{\phi\in\Pi(l)}\left\{\frac{d\int_{-l}^{l}\phi'^2\mathrm{d}x + a\int_{-l}^{l}\phi^2\mathrm{d}x - \frac{G'(0)}{b}\int_{-l}^{l}\int_{-\infty}^{\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x}\right\}\\
&\leq \left.\frac{d\int_{-l}^{l}\phi'^2\mathrm{d}x + a\int_{-l}^{l}\phi^2\mathrm{d}x - \frac{G'(0)}{b}\int_{-l}^{l}\int_{-\infty}^{\infty}K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x}{\int_{-l}^{l}\phi^2\mathrm{d}x}\right|_{\phi=\chi}\\
&= \frac{d\pi^2}{4l^2} + a - \frac{G'(0)}{b}\cdot\frac{\int_{-l}^{l}\int_{-l}^{l}K(x-y)\cos\left(\frac{\pi}{2l}y\right)\mathrm{d}y\cos\left(\frac{\pi}{2l}x\right)\mathrm{d}x}{\int_{-l}^{l}\cos^2\left(\frac{\pi}{2l}x\right)\mathrm{d}x},
\end{aligned}$$

we have $\lim_{l\to+\infty}\gamma_0(l) \leq \lim_{l\to+\infty}P(l) < 0$. Combining this with parts (i) and (ii), we get part (iv). □

Due to the unknown form of $K(x)$, we cannot prove the existence of $\lim_{l\to\infty}P(l)$ easily and calculate it in detail for any $K(x)$ satisfying condition $(K)$. But we can give some specific kernel function $K(x)$ which makes $\lim_{l\to\infty}P(l)$ exist; please see the following remark.

**Remark 3.5.** Since

$$\int_{-l}^{l}\int_{-l}^{l}K(x-y)\cos\left(\frac{\pi}{2l}y\right)\mathrm{d}y\cos\left(\frac{\pi}{2l}x\right)\mathrm{d}x \leq \int_{-l}^{l}\cos^2\left(\frac{\pi}{2l}x\right)\mathrm{d}x,$$

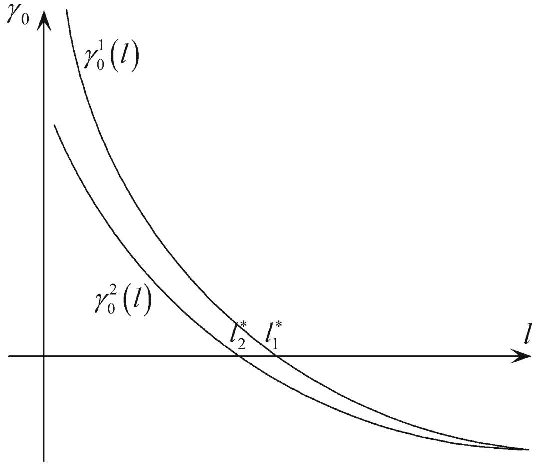


FIG. 1. The effect of nonlocality on $l^*(d)$

$P(l) \geq a - \frac{G'(0)}{b}$. In particular, if $K(x) = \delta(x)$ ($\delta(x)$ is the Dirac delta function), then

$$P(l) \equiv a - \frac{G'(0)}{b}.$$

In this case, problem (1.3) reduces to the problem without nonlocal term studied by Ahn. et al. [1].

For some other specific kernel functions, $\lim_{l\to\infty} P(l)$ can also exist and be equal to $a - \frac{G'(0)}{b}$. For example,

(i) $K(x) = \frac{1}{2L}\mathbf{1}_{[-L,L]}(x)$,

$$\lim_{l\to\infty} P(l) = a - \frac{G'(0)}{b} \lim_{l\to\infty} \frac{2l}{\pi L} \sin \frac{\pi L}{2l} = a - \frac{G'(0)}{b}.$$

(ii) $K(x) = \frac{1}{2\rho}\mathrm{e}^{-\frac{|x|}{\rho}}$,

$$\lim_{l\to\infty} P(l) = a - \frac{G'(0)}{b} \lim_{l\to\infty} \frac{1}{1 + \frac{\pi^2\rho^2}{4l^2}} = a - \frac{G'(0)}{b}.$$

Comparing with the work by Ahn. et al. [1], we show the effect of nonlocality on $l^*(d)$.

**Remark 3.6.** For any fixed $d$, we take $K_1(x) = \frac{1}{2L}\mathbf{1}_{[-L,L]}(x)$ (or $\frac{1}{2\rho}\mathrm{e}^{-\frac{|x|}{\rho}}$) and $K_2(x) = \delta(x)$, for example, and denote by $\gamma_0^1(l)$ and $\gamma_0^2(l)$ the principal eigenvalue of (3.11) with $K(x) = K_1(x)$ and $K(x) = K_2(x)$, respectively. For $K(x) \neq \delta(x)$, since

$$\int_{-l}^{l} \int_{-\infty}^{\infty} K(x-y)\phi(y)\mathrm{d}y\phi(x)\mathrm{d}x < \int_{-l}^{l} \phi^2(x)\mathrm{d}x,$$

it follows from the expression of $\gamma_0$ that $\gamma_0^1(l) > \gamma_0^2(l)$. Hence, $l_1^*(d) > l_2^*(d)$, which means that the nonlocality will increase $l^*(d)$. We use Fig. 1 to show it clearly.

For any fixed $l$, we have the following result.

**Lemma 3.7.** *For any fixed $l$, the following statements are valid:*

(i) *$\gamma_0(d)$ is strictly increasing in $d$.*

(ii) $\lim_{d\to+\infty} \gamma_0(d) > 0$.

(iii) *If* $a - \frac{G'(0)}{b} > 0$, *then* $\gamma_0(d) > 0$ *for all* $d > 0$.
(iv) *If* $P(l) < 0$, *then the equation* $\gamma_0(d) = 0$ *has a unique positive root denoted by* $d^*(l)$. *Furthermore, if* $0 < d < d^*(l)$, *then* $\gamma_0(d) < 0$, *and if* $d > d^*(l)$, *then* $\gamma_0(d) > 0$.

*Proof.* The proof of part (i) is similar to the argument of [2, Lemma 2.2]. Parts (ii)–(iv) can be proved by modifying the arguments of Lemma 3.4. We omit the proof here. □

Similarly to Remark 3.6, we have

**Remark 3.8.** The nonlocality will decrease $d^*(l)$.

The following corollary results directly from Lemmas 3.2, 3.3, 3.4 and 3.7.

**Theorem 3.9.** *The following statements are valid:*
(i) *If* $a - \frac{G'(0)}{b} > 0$, *then* $\lambda_0 > 0$ *for all* $l, d > 0$.
(ii) *If the condition (H) holds, then for any fixed* $d$, *there exists* $l^*(d) > 0$, *such that* $\lambda_0 > 0$ *for* $0 < l < l^*(d)$, $\lambda_0 = 0$ *for* $l = l^*(d)$, *and* $\lambda_0 < 0$ *for* $l > l^*(d)$.
(iii) *If* $P(l) < 0$ *for any fixed* $l$, *then there exists* $d^*(l) > 0$, *such that* $\lambda_0 < 0$ *for* $0 < d < d^*(l)$, $\lambda_0 = 0$ *for* $d = d^*(l)$, *and* $\lambda_0 > 0$ *for* $d > d^*(l)$.

## 4. Spreading and vanishing dichotomy

Before giving the main results, we first give a definition of the spreading and vanishing of the epidemic:
(i) The epidemic *spreads* if

$$h_\infty - g_\infty = \infty \text{ and } \limsup_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) > 0;$$

(ii) The epidemic *vanishes* if

$$h_\infty - g_\infty < \infty \text{ and } \lim_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0.$$

Before giving the vanishing case, we first derive an estimate.

**Lemma 4.1.** *Let* $(u, v, g, h)$ *be the solution of* (1.3). *If* $h_\infty - g_\infty < \infty$, *then there exists a constant* $C > 0$ *such that*

$$\|u(t,\cdot)\|_{C^1([g(t),h(t)])} \le C, \quad \forall\, t > 1. \tag{4.1}$$

*Moreover,*

$$\lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0. \tag{4.2}$$

*Proof.* We can use the method in the proof of [30, Theorem 2.1] to get (4.1). Then, the proof of (4.2) can be done by the method of [34, Theorem 4.1]. □

Next, we give the following result which can be proved by modifying the arguments of [34, Theorem 4.2] (see also [33, Theorem 2.2]).

**Lemma 4.2.** *Let* $d, \mu$ *and* $h_0$ *be positive constants,* $w \in C^{\frac{1+\alpha}{2}, 1+\alpha}([0,\infty) \times [g(t), h(t)])$ *and* $g, h \in C^{1+\frac{\alpha}{2}}([0,\infty))$ *for some* $\alpha > 0$. *We further assume that* $w_0(x) \in \mathscr{X}_1(h_0)$. *If* $(w, g, h)$ *satisfies*

$$\begin{cases} w_t \ge \mathrm{d} w_{xx} - aw, & t > 0,\ g(t) < x < h(t), \\ w(t,x) = 0, & t \ge 0,\ x \le g(t) \text{ or } x \ge h(t), \\ g(0) = -h_0,\ g'(t) \le -\mu w_x(t, g(t)), & t > 0, \\ h(0) = h_0,\ h'(t) \ge -\mu w_x(t, h(t)), & t > 0, \\ w(0,x) = w_0(x) \ge, \not\equiv 0, & -h_0 < x < h_0, \end{cases} \tag{4.3}$$

*and*

$$\lim_{t\to\infty} g(t) = g_\infty > -\infty, \quad \lim_{t\to\infty} g'(t) = 0,$$
$$\lim_{t\to\infty} h(t) = h_\infty < \infty, \quad \lim_{t\to\infty} h'(t) = 0,$$
$$\|w(t,\cdot)\|_{C^1([g(t),h(t)])} \le M, \quad \forall\ t > 1$$

*for some constant $M > 0$. Then,*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} w(t,x) = 0.$$

**Lemma 4.3.** *If $h_\infty - g_\infty < \infty$, then*

$$\lim_{t\to\infty} (\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0.$$

*Proof.* First, we can use the method in the proof of [30, Theorem 2.1] to get

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}([0,\infty)\times[g(t),h(t)])} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,\infty))} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,\infty))} \le C.$$

Recall that $u$ satisfies (4.3). Using Lemmas 4.1 and 4.2, we can get

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

Note that $v(t,x)$ satisfies

$$v_t = -bv + G(u), \quad t > 0, \quad g(t) < x < h(t)$$

and $G(u) \to 0$ uniformly for $x \in [g(t), h(t)]$ as $t \to \infty$; therefore, we have

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0.$$

□

The following result is a sufficient condition such that vanishing occurs.

**Theorem 4.4.** *If $a - \frac{G'(0)}{b} \ge 0$, then $h_\infty - g_\infty < \infty$.*

*Proof.* Direct calculations yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{g(t)}^{h(t)} \left[u(t,x) + \frac{1}{b}\int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y\right]\mathrm{d}x \\
&= \int_{g(t)}^{h(t)} \left[u_t(t,x) + \frac{1}{b}\int_{\mathbb{R}} K(x-y)v_t(t,y)\mathrm{d}y\right]\mathrm{d}x \\
&\quad + h'(t)\left[u(t,h(t)) + \frac{1}{b}\int_{\mathbb{R}} K(h(t)-y)v(t,y)\mathrm{d}y\right] \\
&\quad - g'(t)\left[u(t,g(t)) + \frac{1}{b}\int_{\mathbb{R}} K(g(t)-y)v(t,y)\mathrm{d}y\right] \\
&= \int_{g(t)}^{h(t)} \left\{ \mathrm{d}u_{xx} - au + \int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{b}\int_{\mathbb{R}} K(x-y)[-bv(t,y)+G(u(t,y))]\mathrm{d}y \Bigg\}\,\mathrm{d}x \\
&+h'(t)\frac{1}{b}\int_{\mathbb{R}} K(h(t)-y)v(t,y)\mathrm{d}y - g'(t)\frac{1}{b}\int_{\mathbb{R}} K(g(t)-y)v(t,y)\mathrm{d}y \\
=&-\frac{d}{\mu}(h'(t)-g'(t)) + \int_{g(t)}^{h(t)} \left[-au+\frac{1}{b}\int_{\mathbb{R}} K(x-y)G(u(t,y))\mathrm{d}y\right]\mathrm{d}x \\
&+h'(t)\frac{1}{b}\int_{\mathbb{R}} K(h(t)-y)v(t,y)\mathrm{d}y - g'(t)\frac{1}{b}\int_{\mathbb{R}} K(g(t)-y)v(t,y)\mathrm{d}y.
\end{aligned}
$$

For the ODE system,

$$
\begin{cases}
\frac{\mathrm{d}u}{\mathrm{d}t} = -au+v, & t>0,\\
\frac{\mathrm{d}v}{\mathrm{d}t} = -bv+G(u), & t>0,\\
u(0)=\|u_0\|_{L^\infty},\ v(0)=\|v_0\|_{L^\infty},
\end{cases}
\tag{4.4}
$$

when $G'(0)\le ab$, namely the corresponding basic reproduction number $R_0\le 1$; then, the epidemic always tends to extinction; namely, the solution $(\widetilde{u}(t),\widetilde{v}(t))$ of problem (4.4) tends to $(0,0)$ as $t\to\infty$. By the comparison principle, we have $u(t,x)\le\widetilde{u}(t)$ and $v(t,x)\le\widetilde{v}(t)$. So we can choose $\varepsilon$ small enough and $T_0$ large enough such that $\varepsilon<\frac{bd}{\mu}$ and $v(t,x)<\varepsilon$ for $t\ge T_0$. Integrating from $T_0$ to $t$ gives

$$
\begin{aligned}
&\int_{g(t)}^{h(t)} \left[u(t,x)+\frac{1}{b}\int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y\right]\mathrm{d}x \\
&\le \int_{g(T_0)}^{h(T_0)} \left[u(T_0,x)+\frac{1}{b}\int_{\mathbb{R}} K(x-y)v(T_0,y)\mathrm{d}y\right]\mathrm{d}x \\
&+\left(\frac{d}{\mu}-\frac{\varepsilon}{b}\right)(h(T_0)-g(T_0)) - \left(\frac{d}{\mu}-\frac{\varepsilon}{b}\right)(h(t)-g(t)) \\
&+\int_{T_0}^{t}\int_{g(s)}^{h(s)} \left[-au(s,x)+\frac{1}{b}\int_{\mathbb{R}} K(x-y)G(u(s,y))\mathrm{d}y\right]\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

For all $T_0\le s\le t$, since $\frac{G'(0)}{ab}\le 1$, $\int_{g(s)}^{h(s)} K(x-y)\mathrm{d}x\le 1$ by condition (K) and $u(t,x)=0$ for $x\in\mathbb{R}\backslash(g(t),h(t))$,

$$
\frac{G'(0)}{ab}\int_{\mathbb{R}}\int_{g(s)}^{h(s)} K(x-y)u(s,y)\mathrm{d}x\mathrm{d}y \le \int_{g(s)}^{h(s)} u(s,x)\mathrm{d}x.
$$

By Fubini theorem, we have

$$
\int_{g(s)}^{h(s)} \left[\frac{G'(0)}{ab}\int_{\mathbb{R}} K(x-y)u(s,y)\mathrm{d}y - u(s,x)\right]\mathrm{d}x \le 0.
$$

Since $\frac{G(z)}{z} \le G'(0)$ by the monotonicity of $\frac{G(z)}{z}$, we have

$$\int_{g(s)}^{h(s)} \left[ -au(s,x) + \frac{1}{b} \int_{\mathbb{R}} K(x-y) G(u(s,y)) \mathrm{d}y \right] \mathrm{d}x \le 0.$$

Hence, we have

$$\begin{aligned}\left(\frac{d}{\mu} - \frac{\varepsilon}{b}\right)(h(t) - g(t)) \le & \int_{g(T_0)}^{h(T_0)} \left[ u(T_0,x) + \frac{1}{b} \int_{\mathbb{R}} K(x-y) v(T_0,y) \mathrm{d}y \right] \mathrm{d}x \\ & + \left(\frac{d}{\mu} - \frac{\varepsilon}{b}\right)(h(T_0) - g(T_0));\end{aligned}$$

then, we can get that $h_\infty - g_\infty < \infty$ by letting $\varepsilon \to 0$ and $t \to \infty$. □

**Theorem 4.5.** *Assume that the condition (H) holds. If $h_0 \ge l^*(d)$, then $h_\infty = -g_\infty = \infty$ and spreading occurs.*

*Proof.* We first consider the case that $h_0 > l^*(d)$. Denote by $\lambda_0$ and $(\phi, \psi)$ the principal eigenvalue and the corresponding eigenfunction of problem (3.3) with $l = h_0$, respectively, where $(\phi, \psi) = (\theta\psi, \psi) \gg 0$ in $(-h_0, h_0)$ and $\|\phi\|_{L^\infty} = 1$. It follows from Theorem 3.9 that $\lambda_0 < 0$.

Now, we construct a suitable lower solution of problem (1.3). Define

$$\underline{u}(t,x) = \begin{cases} \epsilon\phi(x), & t \ge 0,\ -h_0 \le x \le h_0, \\ 0, & t \ge 0,\ x < -h_0 \text{ or } x > h_0, \end{cases}$$
$$\underline{v}(t,x) = \begin{cases} \epsilon\psi(x), & t \ge 0,\ -h_0 \le x \le h_0, \\ 0, & t \ge 0,\ x < -h_0 \text{ or } x > h_0, \end{cases}$$

where positive constant $\epsilon$ will be selected later.

Direct computations show that

$$\begin{aligned} & \underline{u}_t - d\underline{u}_{xx} + a\underline{u} - \int_{\mathbb{R}} K(x-y)\underline{v}(t,y)\mathrm{d}y \\ & = -d\epsilon\phi'' + a\epsilon\phi - \epsilon \int_{\mathbb{R}} K(x-y)\psi(y)\mathrm{d}y \\ & = \epsilon\lambda_0\phi \le 0, \end{aligned}$$

and

$$\begin{aligned} \underline{v}_t + b\underline{v} - G(\underline{u}) &= b\epsilon\psi - G'(\xi)\underline{u} \\ &= \epsilon\lambda_0\psi + (G'(0) - G'(\xi))\epsilon\theta\psi \\ &= \epsilon\psi[\lambda_0 + \theta(G'(0) - G'(\xi))], \end{aligned}$$

for all $t > 0$ and $-h_0 < x < h_0$, where $\xi(t,x) \in (0, \underline{u})$. Noting that $\lambda_0 < 0$ and $0 < \xi(t,x) < \underline{u}(t,x) \le \epsilon$, we can choose $\epsilon$ small enough such that

$$\lambda_0 + \theta(G'(0) - G'(\xi)) < 0$$

and

$$\epsilon\phi(x) \le u_0(x),\ \epsilon\psi(x) \le v_0(x) \text{ for } \forall\ x \in [-h_0, h_0].$$

Hence,

$$\begin{cases} \underline{u}_t \le d\underline{u}_{xx} - a\underline{u} + \int_{\mathbb{R}} K(x-y)\underline{v}(t,y)\mathrm{d}y, & t>0,\ -h_0<x<h_0, \\ \underline{v}_t \le -b\underline{v} + G(\underline{u}), & t>0,\ -h_0<x<h_0, \\ \underline{u}(t,x) = \underline{v}(t,x) = 0, & t\ge 0,\ x\le -h_0 \text{ or } \ x\ge h_0, \\ -h_0' = 0 \ge -\mu \underline{u}_x(t,-h_0), & t>0, \\ h_0' = 0 \le -\mu \underline{u}_x(t,h_0), & t>0, \\ \underline{u}(0,x) \le u_0(x),\ \underline{v}(0,x)\le v_0(x), & -h_0<x<h_0. \end{cases} \tag{4.5}$$

Thus, Remark 2.3 implies that $u(t,x)\ge \underline{u}(t,x)$ and $v(t,x)\ge \underline{v}(t,x)$ in $[0,\infty)\times[-h_0,h_0]$. It follows that

$$\begin{aligned} &\liminf_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) \\ &\quad \ge \liminf_{t\to\infty}(\|u(t,\cdot)\|_{C([-h_0,h_0])} + \|v(t,\cdot)\|_{C([-h_0,h_0])}) \\ &\quad \ge \epsilon(\|\phi(\cdot)\|_{C([-h_0,h_0])} + \|\psi(\cdot)\|_{C([-h_0,h_0])}) > 0, \end{aligned}$$

By Lemma 4.3, we know that $h_\infty - g_\infty = +\infty$.

If $h_0 = l^*(d)$, then for any time $t_0$, we have $h(t_0) > l^*(d)$. Replacing the initial time 0 by the time $t_0$, we can derive that $h_\infty - g_\infty = +\infty$ as above. □

**Theorem 4.6.** *Assume that the condition (H) holds. If $h_0 < l^*(d)$ and*

$$\|u_0(x)\|_{C([-h_0,h_0])}, \quad \|v_0(x)\|_{C([-h_0,h_0])}$$

*are sufficiently small, then $h_\infty - g_\infty < \infty$ and*

$$\lim_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0.$$

*Proof.* We construct a suitable upper solution. Denote by $\lambda_0$ and $(\phi,\psi)$ the principal eigenvalue and the corresponding eigenfunction of problem (3.3) with $l = h_0$, respectively, where $(\phi,\psi) = (\theta\psi,\psi) \gg 0$ in $(-h_0,h_0)$ and $\|\phi\|_{L^\infty} = 1$. Since $h_0 < l^*(d)$, it follows from Theorem 3.9 that $\lambda_0 > 0$.

We set

$$\begin{aligned} &\sigma(t) = h_0\left(1+\delta-\frac{\delta}{2}\mathrm{e}^{-\delta t}\right), \quad t\ge 0, \\ &\overline{u}(t,x) = \begin{cases} \varepsilon \mathrm{e}^{-\delta t}\phi\left(\frac{xh_0}{\sigma(t)}\right), & t\ge 0,\ -\sigma(t)\le x\le \sigma(t), \\ 0, & t\ge 0,\ x<-\sigma(t) \text{ or } x>\sigma(t), \end{cases} \\ &\overline{v}(t,x) = \begin{cases} \varepsilon \left(\frac{h_0}{\sigma(t)}\right)^2 \mathrm{e}^{-\delta t}\psi\left(\frac{xh_0}{\sigma(t)}\right), & t\ge 0,\ -\sigma(t)\le x\le \sigma(t), \\ 0, & t\ge 0,\ x<-\sigma(t) \text{ or } x>\sigma(t), \end{cases} \end{aligned}$$

where positive constant $\varepsilon$ and $\delta$ will be selected later. Applying [16, Lemma 2.35] to problem (3.4), one can easily see that $x\phi'(x)\le 0$ for $-h_0\le x\le h_0$. By the direct computations, we have

$$\begin{aligned} &\overline{u}_t - d\overline{u}_{xx} + a\overline{u} - \int_{\mathbb{R}} K(x-y)\overline{v}(t,y)\mathrm{d}y \\ &\quad = -\delta\overline{u} - \varepsilon\mathrm{e}^{-\delta t}\phi'\frac{xh_0\sigma'}{\sigma^2} - \varepsilon\mathrm{e}^{-\delta t}d\phi''\left(\frac{h_0}{\sigma}\right)^2 + a\overline{u} \\ &\qquad - \int_{\mathbb{R}} K(x-y)\overline{v}(t,y)\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\geq \varepsilon \mathrm{e}^{-\delta t}\left\{-\delta\phi+\left(\frac{h_0}{\sigma}\right)^2\left[-a\phi+\int_{\mathbb{R}} K(x-y)\psi\left(\frac{yh_0}{\sigma(t)}\right)\mathrm{d}y+\lambda_0\phi\right]\right.\\
&\qquad\left.+a\phi-\left(\frac{h_0}{\sigma}\right)^2\int_{\mathbb{R}} K(x-y)\psi\left(\frac{yh_0}{\sigma(t)}\right)\mathrm{d}y\right\}\\
&=\varepsilon \mathrm{e}^{-\delta t}\left\{-\delta\phi+\left(\frac{h_0}{\sigma}\right)^2\lambda_0\phi+\left[1-\left(\frac{h_0}{\sigma}\right)^2\right]a\phi\right\}\\
&>\varepsilon \mathrm{e}^{-\delta t}\phi\left\{-\delta+\frac{1}{\left(1+\delta-\frac{\delta}{2}\mathrm{e}^{-\delta t}\right)^2}\lambda_0+\left[1-\frac{1}{\left(1+\delta-\frac{\delta}{2}\mathrm{e}^{-\delta t}\right)^2}\right]a\right\}.
\end{aligned}
$$

For any $-\sigma(t) < x < \sigma(t)$, since $\lambda_0 > 0$, we can always find some $\delta_1$ sufficiently small such that

$$
\frac{1}{(1+\delta)^2}\lambda_0 \geq \delta;
$$

then,

$$
\overline{u}_t - d\overline{u}_{xx} + a\overline{u} - \int_{\mathbb{R}} K(x-y)\overline{v}(t,y)\mathrm{d}y > \varepsilon \mathrm{e}^{-\delta t}\phi\left[-\delta+\frac{1}{(1+\delta)^2}\lambda_0\right] \geq 0,
$$

for all $t > 0$ and $-\sigma(t) < x < \sigma(t)$.

Moreover,

$$
\begin{aligned}
\overline{v}_t + b\overline{v} - G(\overline{u}) &= -\frac{2\sigma'(t)}{\sigma(t)}\overline{v} - \delta\overline{v} - \left(\frac{h_0}{\sigma}\right)^2 \varepsilon \mathrm{e}^{-\delta t}\psi'\frac{xh_0\sigma'}{\sigma^2} + b\left(\frac{h_0}{\sigma}\right)^2\varepsilon \mathrm{e}^{-\delta t}\psi - G(\overline{u})\\
&\geq -\frac{2\sigma'(t)}{\sigma(t)}\overline{v} - \delta\overline{v} + \left(\frac{h_0}{\sigma}\right)^2 \varepsilon \mathrm{e}^{-\delta t}(\lambda_0\psi + G'(0)\phi) - G'(\xi)\overline{u}\\
&= \overline{v}\left\{-\frac{\delta^2 \mathrm{e}^{-\delta t}}{1+\delta-\frac{\delta}{2}\mathrm{e}^{-\delta t}} - \delta + \lambda_0 + \left[G'(0) - \left(1+\delta-\frac{\delta}{2}\mathrm{e}^{-\delta t}\right)^2 G'(\xi)\right]\theta\right\},
\end{aligned}
$$

for all $t > 0$ and $-\sigma(t) < x < \sigma(t)$, where $\xi \in (0, \overline{u})$. Since $\overline{u} \leq \varepsilon$, we can choose $\delta_2$ and $\varepsilon$ sufficiently small such that

$$
-\delta^2 - \delta + \lambda_0 + \left[G'(0) - (1+\delta)^2 G'(\xi)\right]\theta \geq 0
$$

and

$$
\varepsilon \leq -\frac{\delta^2 h_0(1+\delta)}{2\mu\phi'(h_0)}.
$$

Then,

$$
\overline{v}_t + b\overline{v} - G(\overline{u}) \geq 0.
$$

On the other hand, we have that

$$
\sigma'(t) = h_0\frac{\delta^2}{2}\mathrm{e}^{-\delta t} \geq\ -\mu\varepsilon\frac{h_0}{\sigma(t)}\phi'(h_0)\mathrm{e}^{-\delta t} =\ -\mu\overline{u}_x(t,\sigma(t)),
$$

and

$$
\begin{aligned}
-\sigma'(t) = -h_0\frac{\delta^2}{2}\mathrm{e}^{-\delta t} \leq\ \mu\varepsilon\frac{h_0}{\sigma(t)}\phi'(h_0)\mathrm{e}^{-\delta t} &=\ -\mu\varepsilon\frac{h_0}{\sigma(t)}\phi'(-h_0)\mathrm{e}^{-\delta t}\\
&=\ -\mu\overline{u}_x(t,-\sigma(t)).
\end{aligned}
$$

We take $\delta = \min\{\delta_1, \delta_2\}$. For this $\delta$ and above $\varepsilon$, if $u_0$ and $v_0$ are small enough such that

$$\|u_0\|_{C([-h_0,h_0])} \le \varepsilon\phi\left(\frac{x}{1+\frac{\delta}{2}}\right) \quad \text{and} \quad \|v_0\|_{C([-h_0,h_0])} \le \varepsilon\left(\frac{1}{1+\frac{\delta}{2}}\right)^2 \psi\left(\frac{x}{1+\frac{\delta}{2}}\right),$$

then

$$u_0(x) \le \overline{u}(0,x), \quad v_0(x) \le \overline{v}(0,x) \quad \text{for} \quad x \in [-h_0, h_0].$$

Thus, it follows from Theorem 2.2 that $g(t) \ge -\sigma(t)$ and $h(t) \le \sigma(t)$ for all $t \ge 0$. Then, $h_\infty - g_\infty \le \lim_{t\to\infty} 2\sigma(t) = 2h_0(1+\delta) < \infty$. ☐

**Theorem 4.7.** *Assume that the condition (H) holds. If $h_0 < l^*(d)$ and*

$$\|u_0(x)\|_{C([-h_0,h_0])}, \quad \|v_0(x)\|_{C([-h_0,h_0])}$$

*are sufficiently large, then $h_\infty - g_\infty = \infty$ and*

$$\liminf_{t\to\infty} \left(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}\right) > 0.$$

*Proof.* We first note that Theorem 3.9 implies that there exists $T^* > 0$ such that $\lambda_0(d, \sqrt{T^*}) < 0$.

Inspired by the argument of [12, Proposition 5.3] (see also [1, Theorem 4.2]), let $\nu_0$ be the eigenvalue of

$$\begin{cases} -d\varphi'' - \frac{\operatorname{sgn}(x)}{2}\varphi' = \nu_0\varphi, & -1 < x < 1, \\ \varphi(-1) = \varphi(1) = 0, \end{cases}$$

the corresponding function $\varphi > 0$ and $\|\varphi\|_{L^\infty(-1,1)} = 1$.

Now we construct a suitable lower solution to (1.3). Define

$$\begin{aligned}
&\eta(t) = \sqrt{t+\varrho}, \ t \ge 0, \\
&\underline{u}(t,x) = \begin{cases} \frac{m}{(t+\varrho)^k}\varphi\left(\frac{x}{\sqrt{t+\varrho}}\right), & t \ge 0, \ -\eta(t) \le x \le \eta(t), \\ 0, & t \ge 0, \ |x| > \eta(t), \end{cases} \\
&\underline{v}(0,x) = \begin{cases} v_0(x), & -\sqrt{\varrho} \le x \le \sqrt{\varrho}, \\ 0, & |x| > \sqrt{\varrho}, \end{cases} \\
&\underline{v}(t,x) = \mathrm{e}^{-bt}\left(\int_0^t \mathrm{e}^{b\tau} G(\underline{u}(\tau,x))d\tau + \underline{v}(0,x)\right), \quad t \ge 0, \quad -\eta(t) \le x \le \eta(t),
\end{aligned}$$

where the constants $\varrho, \ m, \ k$ are chosen as follows:

$$0 < \varrho \le \min\{1, h_0^2\}, \quad k \ge \nu_0 + a(T^*+1), \quad m \ge \frac{(T^*+1)^k}{-2\mu\varphi'(1)}.$$

Direct computations yield

$$\begin{aligned}
&\underline{u}_t - d\underline{u}_{xx} + a\underline{u} - \int_{g(t)}^{h(t)} K(x-y)\underline{v}(t,y)\mathrm{d}y \\
&\le -\frac{m}{(t+\varrho)^{k+1}}\left[k\varphi + \frac{x}{2\sqrt{t+\varrho}}\varphi' + d\varphi'' - a(t+\varrho)\varphi\right] \\
&\le -\frac{m}{(t+\varrho)^{k+1}}\left[k\varphi + \frac{\operatorname{sgn}(x)}{2}\varphi' + d\varphi'' - a(T^*+1)\varphi\right]
\end{aligned}$$

$$\leq -\frac{m}{(t+\varrho)^{k+1}}\left[d\varphi'' + \frac{\operatorname{sgn}(x)}{2}\varphi' + \nu_0\varphi\right]$$
$$= 0,$$

and

$$\underline{v}_t + b\underline{v} - G(\underline{u}) = 0 \quad \text{for all} \quad 0 < t \leq T^* \quad \text{and} \quad -\eta(t) < x < \eta(t).$$

Moreover,

$$\eta'(t) + \mu\underline{u}_x(t,\eta(t)) = \frac{1}{2\sqrt{t+\varrho}} + \frac{\mu m}{(t+\varrho)^{k+\frac{1}{2}}}\varphi'(1) \leq 0 \quad \text{for} \quad 0 < t \leq T^*.$$

If

$$\underline{u}(0,x) = \frac{m}{\varrho^k}\varphi\left(\frac{x}{\sqrt{\varrho}}\right) \leq u_0(x) \text{ in } [-\sqrt{\varrho},\sqrt{\varrho}],$$

then we have

$$\begin{cases} \underline{u}_t \leq d\underline{u}_{xx} - a\underline{u} + \int_{\mathbb{R}} K(x-y)\underline{v}(t,y)\mathrm{d}y, & 0 < t \leq T^*,\ -\eta(t) < x < \eta(t), \\ \underline{v}_t = -b\underline{v} + G(\underline{u}), & 0 < t \leq T^*,\ -\eta(t) < x < \eta(t), \\ \underline{u}(t,x) = \underline{v}(t,x) = 0, & 0 \leq t \leq T^*,\ x \leq -\eta(t) \text{ or } \ x \geq \eta(t), \\ -\eta'(t) \geq -\mu\underline{u}_x(t,-\eta(t)), & 0 < t \leq T^*, \\ \eta'(t) \leq -\mu\underline{u}_x(t,\eta(t)), & 0 < t \leq T^*, \\ \underline{u}(0,x) \leq u_0(x),\ \underline{v}(0,x) = v_0(x), & -\eta(0) < x < \eta(0). \end{cases}$$

Noting that $\eta(0) = \sqrt{\varrho} \leq h_0$, we can use Remark 2.3 to conclude that $h(t) \geq \eta(t)$ and $g(t) \leq -\eta(t)$ in $[0,T^*]$. Specially, we obtain

$$h(T^*) \geq \eta(T^*) = \sqrt{T^*+\varrho} > \sqrt{T^*}$$

and $g(T^*) < -\sqrt{T^*}$. Then,

$$(-l^*,l^*) \subseteq (-\sqrt{T^*},\sqrt{T^*}) \subseteq (g(t),h(t)).$$

Hence, we have $h_\infty - g_\infty = +\infty$ by Theorem 4.5. □

The following theorem is a direct result of Theorems 4.5, 4.6 and 4.7.

**Theorem 4.8.** *Assume that the condition (H) holds. For some $\sigma > 0$ and any given $(\omega_1(x),\omega_2(x)) \in \mathscr{X}_1(h_0) \times \mathscr{X}_2(h_0)$, let $(u,v,g,h)$ be a solution to problem (1.3) with $(u_0(x),v_0(x)) = \sigma(\omega_1,\omega_2)$, then there exists $\sigma^* \geq 0$ such that spreading occurs if $\sigma > \sigma^*$, and vanishing occurs if $0 < \sigma \leq \sigma^*$. Moreover, $\sigma^* = 0$ if $h_0 \geq l^*(d)$, and $\sigma^* > 0$ if $h_0 < l^*(d)$ for some fixed $d$.*

*Proof.* For the reason why $\sigma^*$ exists, the readers can refer to [19, Theorem 5.7]. Here we omit the details for brevity. □

Similarly, we have the following result.

**Theorem 4.9.** *Assume that $P(h_0) < 0$. The conclusions except the last one in Theorem 4.8 still hold. For $\sigma^*$ in Theorem 4.8, $\sigma^* = 0$ if $0 < d \leq d^*(h_0)$, and $\sigma^* > 0$ if $d > d^*(h_0)$.*

When spreading happens, we have the following lemma:

**Lemma 4.10.** *Assume that the condition (H) holds. If spreading occurs, then*

$$\limsup_{t\to\infty}(u(t,x),v(t,x)) \leq (u^*,v^*)$$

*uniformly for* $x \in \mathbb{R}$, *where* $(u^*, v^*)$ *is the unique positive equilibrium of*

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = -au(t) + v(t), & t > 0, \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} = -bv(t) + G(u(t)), & t > 0, \\ u(0) = \|u_0\|_{L^\infty([-h_0,h_0])}, \ v(0) = \|v_0\|_{L^\infty([-h_0,h_0])}. \end{cases} \tag{4.6}$$

*Proof.* We denote by $(u(t), v(t))$ the solution of (4.6). Applying the comparison principle gives

$$(u(t,x), v(t,x)) \le (u(t), v(t))$$

for $t > 0$ and $g(t) \le x \le h(t)$. Since $P(l) \ge a - \frac{G'(0)}{b}$ for any $l$, the condition (H) implies that $a - \frac{G'(0)}{b} < 0$, namely the basic reproduction number $R_0 (= \frac{G'(0)}{ab}) > 1$ for problem (4.6). Hence, problem (4.6) has a unique positive equilibrium denoted by $(u^*, v^*)$ and $\lim_{t\to\infty}(u(t), v(t)) = (u^*, v^*)$. Hence,

$$\limsup_{t\to\infty}(u(t,x), v(t,x)) \le (u^*, v^*)$$

uniformly for $x \in \mathbb{R}$. □

Combining Lemmas 4.3 and 4.10, we immediately have the following spreading–vanishing dichotomy theorem.

**Theorem 4.11.** *Let* $(u, v, g, h)$ *be the solution of the free boundary problem* (1.3). *Assume that the condition (H) holds. Then, the following alternative holds:*

(i) *Spreading:* $h_\infty - g_\infty = \infty$ *and*

$$(0,0) < \liminf_{t\to\infty}(u(t,x), v(t,x)) \le \limsup_{t\to\infty}(u(t,x), v(t,x)) \le (u^*, v^*)$$

*uniformly for* $x \in \mathbb{R}$;

(ii) *Vanishing:* $h_\infty - g_\infty < \infty$ *and* $\lim_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0$.

## 5. Spreading speed

When the spreading happens, we give some rough estimates of the asymptotic spreading speed in this section. In particular, if $a$ and $b$ are small enough and the condition (H) holds, then the spreading will happen for some suitable initial value $u_0$, $v_0$ and $h_0$. Similarly to the argument of [11,28], we can provide an upper bound for $\limsup_{t\to\infty} \frac{h(t)}{t}$ and $\limsup_{t\to\infty} \frac{-g(t)}{t}$, which shows that the asymptotic spreading speed (if exists) for problem (1.3) cannot be faster than the minimal speed of traveling wave fronts of

$$\begin{cases} u_t = \mathrm{d}u_{xx} - au + \int_{\mathbb{R}} K(x-y)v(t,y)\mathrm{d}y, & t > 0, \ x \in \mathbb{R}, \\ v_t = -bv + G(u), & t > 0, \ x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Xu and Zhao [37] proved that there exists $c^* > 0$ such that for any $c \ge c^*$, the following problem

$$\begin{cases} -cU'(\xi) = \mathrm{d}U''(\xi) - aU(\xi) + \int_{\mathbb{R}} K(y)V(\xi - y)\mathrm{d}y, & \xi \in \mathbb{R}, \\ -cV'(\xi) = -bV(\xi) + G(U(\xi)), & \xi \in \mathbb{R}, \\ (U,V)(-\infty) = (u^*, v^*), \ (U,V)(+\infty) = (0,0), & \\ U'(\xi) < 0, \ V'(\xi) < 0, & \xi \in \mathbb{R} \end{cases} \tag{5.2}$$

has a solution $(U(\xi), V(\xi))$ with $\xi = x - ct$, while problem (5.2) has no solution for $c < c^*$. $c^*$ is called the minimal speed of the traveling waves of (5.1).

**Theorem 5.1.** *Let $(u, v, g, h)$ be a solution of (1.3). When spreading occurs, if $a$ and $b$ are sufficiently small and*

$$\max_{x\in[-h_0,h_0]} u_0(x) < u^*, \qquad \max_{x\in[-h_0,h_0]} v_0(x) < v^*, \tag{5.3}$$

*then we have*

$$\limsup_{t\to\infty} \frac{h(t)}{t}, \quad \limsup_{t\to\infty} \frac{-g(t)}{t} \le c^*.$$

*Proof.* We will construct a suitable upper solution to (1.3) and then apply Theorem 2.2.

In (5.1), we replace $a$ by $\widetilde{a} = a - \varepsilon_1$ and replace $b$ by $\widetilde{b} = b - \varepsilon_2$, where $\varepsilon_1$ and $\varepsilon_2$ are small positive constants. We then denote by $\widetilde{c}^*$ the minimal speed of the traveling waves to the modified problem of (5.1). We thus have

$$\begin{cases} -\widetilde{c}^* U'(\xi) = \mathrm{d}U''(\xi) - \widetilde{a}U(\xi) + \int_{\mathbb{R}} K(y)V(\xi - y)\mathrm{d}y, & \xi \in \mathbb{R}, \\ -\widetilde{c}^* V'(\xi) = -\widetilde{b}V(\xi) + G(U(\xi)), & \xi \in \mathbb{R}, \\ (U, V)(-\infty) = (\widetilde{u}^*, \widetilde{v}^*),\ (U, V)(+\infty) = (0, 0), & \\ U'(\xi) < 0,\ V'(\xi) < 0, & \xi \in \mathbb{R}, \end{cases}$$

where $(\widetilde{u}^*, \widetilde{v}^*) = (u^* + \varepsilon_3, v^* + \varepsilon_4)$, $(\varepsilon_3, \varepsilon_4) \to (0, 0)$ as $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.

Let $\vartheta(x)$ be a smooth function satisfying

$$\vartheta(x) \in [0, 1],\ \vartheta(x) = 0 \quad \text{for} \quad x \le -1,\ \vartheta(x) = 1 \quad \text{for} \quad x \ge 0, \quad \vartheta'(x) \ge 0.$$

Since $U'(\xi) < 0,\ V'(\xi) < 0$ and $\lim_{\xi\to+\infty}(U, V)(\xi) = (0, 0)$, we can find $L > h_0$ large enough such that

$$\widetilde{c}^* \ge -\mu U'(L) \quad \text{and} \quad \varepsilon_1 + \frac{1}{L}\left(\vartheta'\widetilde{c}^* - \frac{d\vartheta''}{L}\right) > 0.$$

We now define

$$\overline{h}(t) = R + \widetilde{c}^* t, \quad \overline{g}(t) = -R - \widetilde{c}^* t, \quad t \ge 0,$$

$$\overline{u}(t, x) = \begin{cases} U(x - \widetilde{c}^* t - R + L) - \vartheta\left(\frac{x - \widetilde{c}^* t - R + 1}{L}\right) U(L), & t \ge 0,\ 0 \le x \le \overline{h}(t), \\ U(-x - \widetilde{c}^* t - R + L) - \vartheta\left(\frac{-x - \widetilde{c}^* t - R + 1}{L}\right) U(L), & t \ge 0,\ \overline{g}(t) \le x < 0, \end{cases}$$

$$\overline{v}(t, x) = \begin{cases} V(x - \widetilde{c}^* t - R + L) - \vartheta\left(\frac{x - \widetilde{c}^* t - R + 1}{L}\right) V(L), & t \ge 0,\ 0 \le x \le \overline{h}(t), \\ V(-x - \widetilde{c}^* t - R + L) - \vartheta\left(\frac{-x - \widetilde{c}^* t - R + 1}{L}\right) V(L), & t \ge 0,\ \overline{g}(t) \le x < 0, \end{cases}$$

where $R > L$ will be determined later.

For given $\vartheta(x)$, if $a$ and $b$ are small enough such that

$$a < \frac{V(L)}{U(L)} \inf_{x\in(-1,\infty)} \left\{\frac{\int_{\mathbb{R}} K(x - y)\vartheta(y)\mathrm{d}y}{\vartheta(x)}\right\} \quad \text{and} \quad b < \frac{U(L)}{V(L)} \min_{U_\theta\in(0,\widetilde{u}^*)} G'(U_\theta), \tag{5.4}$$

then for all $t > 0$ and $0 \le x < \overline{h}(t)$, direct computations yield

$$\begin{aligned} &\overline{u}_t - d\overline{u}_{xx} + a\overline{u} - \int_{\mathbb{R}} K(x - y)\overline{v}(t, y)\mathrm{d}y \\ &= -\widetilde{c}^* U' + \frac{\vartheta'\widetilde{c}^*}{L} U(L) - \mathrm{d}U'' - \frac{d\vartheta''}{L^2} U(L) \\ &\quad + a(U - \vartheta U(L)) - \int_{\mathbb{R}} K(x - y)(V - \vartheta V(L))\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_1 U + \frac{\vartheta' \widetilde{c}^*}{L} U(L) - \frac{d\vartheta''}{L^2} U(L) - a\vartheta U(L) + \int_{\mathbb{R}} K(x-y)\vartheta V(L)\mathrm{d}y \\
&= \left(\varepsilon_1 + \frac{\vartheta' \widetilde{c}^*}{L} - \frac{d\vartheta''}{L^2}\right) U(L) - a\vartheta U(L) + \int_{\mathbb{R}} K(x-y)\vartheta \mathrm{d}y \cdot V(L) \\
&> 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{v}_t + b\overline{v} - G(\overline{u}) =& -\widetilde{c}^* V' + \frac{\vartheta' \widetilde{c}^*}{L} V(L) + b(V - \vartheta V(L)) - G(U - \vartheta U(L)) \\
=& \, \varepsilon_2 V + \frac{\vartheta' \widetilde{c}^*}{L} V(L) - b\vartheta V(L) + G(U) - G(U - \vartheta U(L)) \\
>& - b\vartheta V(L) + G'(U_\xi)\vartheta U(L) \\
\geq& - b\vartheta V(L) + \min_{U_\theta \in (0,\widetilde{u}^*)} G'(U_\theta)\vartheta U(L) \\
>& \, 0,
\end{aligned}
$$

where $U_\xi \in (U - \vartheta U(L), U) \subset (0, \widetilde{u}^*)$. Similarly, for all $t > 0$ and $\overline{g}(t) < x < 0$, we have

$$
\overline{u}_t - d\overline{u}_{xx} + a\overline{u} - \int_{\mathbb{R}} K(x-y)\overline{v}(t,y)\mathrm{d}y \geq 0,
$$

and

$$
\overline{v}_t + b\overline{v} - G(\overline{u}) \geq 0.
$$

Clearly, $\overline{h}(0) = R > h_0,\ \overline{g}(0) = -R < -h_0$ and

$$
\overline{h}'(t) = \widetilde{c}^* \geq -\mu U'(L) = -\mu \overline{u}_x(t, \overline{h}(t)), \quad \overline{g}'(t) = -\widetilde{c}^* \leq \mu U'(L) = -\mu \overline{u}_x(t, \overline{g}(t)).
$$

Since $(U,V)(-\infty) = (\widetilde{u}^*, \widetilde{v}^*) > (u^*, v^*)$, we can choose $R \geq L + h_0 + 1$ large enough such that

$$
U(h_0 - R + L) > u^* \quad \text{and} \quad V(h_0 - R + L) > v^*.
$$

For this $R$, it follows from the definition of $\vartheta(x)$ that

$$
\vartheta\left(\frac{x - R + 1}{L}\right) = \vartheta\left(\frac{-x - R + 1}{L}\right) = 0 \quad \text{for} \quad x \in [-h_0, h_0].
$$

By the condition

$$
\max_{x \in [-h_0, h_0]} u_0(x) < u^* \quad \text{and} \quad \max_{x \in [-h_0, h_0]} v_0(x) < v^*,
$$

we have

$$
\overline{u}(0,x) = U(x - R + L) > U(h_0 - R + L) > u^* > u_0(x),
$$

and

$$
\overline{v}(0,x) = V(x - R + L) > V(h_0 - R + L) > v^* > v_0(x),
$$

for $-h_0 \leq x \leq h_0$. Finally, we note that

$$
\overline{u}(t, \overline{h}(t)) = \overline{u}(t, \overline{g}(t)) = \overline{v}(t, \overline{h}(t)) = \overline{v}(t, \overline{g}(t)) = 0.
$$

Hence, we can apply the comparison principle to conclude that $h(t) \leq \overline{h}(t)$ and $g(t) \geq \overline{g}(t)$. It follows that

$$
\limsup_{t \to \infty} \frac{h(t)}{t}, \quad \limsup_{t \to \infty} \frac{-g(t)}{t} \leq \widetilde{c}^*.
$$

Due to the continuous dependence of $\widetilde{c}^*$ on the parameters, the desired result follows by letting $\varepsilon_1,\ \varepsilon_2 \to 0$. □

**Remark 5.2.** Because it is hard to describe $\liminf_{t\to\infty} \frac{h(t)}{t}$ and $\liminf_{t\to\infty} \frac{-g(t)}{t}$ precisely and further prove

$$\liminf_{t\to\infty} \frac{h(t)}{t} = \limsup_{t\to\infty} \frac{h(t)}{t} \quad \text{and} \quad \liminf_{t\to\infty} \frac{-g(t)}{t} = \limsup_{t\to\infty} \frac{-g(t)}{t},$$

we cannot give the estimates of the asymptotic spreading speed, namely the value of $\lim_{t\to\infty} \frac{h(t)}{t}$ and $\lim_{t\to\infty} \frac{-g(t)}{t}$. But if $\lim_{t\to\infty} \frac{h(t)}{t}$ and $\lim_{t\to\infty} \frac{-g(t)}{t}$ exist, it follows from the property of the limit and Theorem 5.1 that

$$\lim_{t\to\infty} \frac{h(t)}{t} \le \widetilde{c}^* \quad \text{and} \quad \lim_{t\to\infty} \frac{-g(t)}{t} \le \widetilde{c}^*.$$

Hence, if spreading occurs, the asymptotic spreading speed of the front (if exist) will not be faster than the minimal speed of corresponding traveling wave.

## 6. Discussion

Comparing with the problem without the nonlocal term, we find it is difficult to consider the principal eigenvalue of problem (3.1), which is mainly due to the existence of the nonlocal term. As in [1,26], we should build the relationship between the principal eigenvalue $\lambda_0$ of problem (3.4) and the principal eigenvalue $\gamma_0$ of problem (3.11). By using $\Theta(d, l)$ as a bridge, we derive that $\gamma_0$ has the same sign as $\lambda_0$. Then, we only need to study the sign of $\gamma_0$.

The main results show that the nonlocal term has an influence on the spreading of the epidemic disease. For the problem without nonlocal term, the results in [26] indicated that vanishing will occur if $a - \frac{G'(0)}{b} \ge 0$. While for $a - \frac{G'(0)}{b} < 0$, there exists $\widehat{d}$ such that spreading always occurs if $0 < d \le \widehat{d}$, and whether spreading occurs or not depends on the initial data for $d > \widehat{d}$. In fact, this special case is exactly $K(x) = \delta(x)$. For problem (1.3) with nonlocal term, we deduce the similar results; see Theorems 4.4 and 4.9. But it follows from Remark 3.8 that the nonlocal term will decrease $d^*$.

Theorem 4.8 illustrates that for some fixed diffusion rate $d$, the epidemic will spread if condition (H) holds and the initial habit is large enough. While if the initial habit is small but the initial densities of the infectious agents and infective human population are large enough, the epidemic will also spread. Otherwise, vanishing will happen.

In Sect. 5, we consider a specific situation; namely, $a$ and $b$ are small enough such that (5.4) holds; in addition, $u_0$ and $v_0$ are suitably small such that (5.3) holds. Under this specific situation, we estimate roughly the asymptotic spreading speed when spreading happens. Here, we should emphasis that these conditions are added only for the convenience of constructing a suitable upper solution. For a two species system with free boundaries, Du et al. [13] firstly gave the precise asymptotic spreading speed determined by a certain traveling wave type system of one space dimension, which is called a semi-wave. For the precise asymptotic spreading speed of problem (1.3), due to the complexities and difficulties arising in constructing suitable upper–lower solutions precisely, we leave it for further consideration.

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