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Phase transition in information propagation on high-order networks

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The phase transition in information spreading was investigated. Two phase transitions were studied, respectively, from the point of view of network information and the node information, and the corresponding simulations were implemented on high-order networks. In order to explore the phase transition in information dissemination, this paper analyzes the characteristics of social networks and constructs a high-order network model with explicit, subexplicit, and implicit relationships. The factors affecting interpersonal communication were analyzed, the social cumulative effect was introduced, and two transmission models with characteristics of permanent forgetting and temporary forgetting were proposed. The comparison between simulated experiment and real case shows that the network mode accord with the characteristics of the actual network, and the two models of phase transitions are closer to the actual.

Keywords: High-order network; information propagation; SCE; SI2R model; phase transition.

PACS numbers: 64.60.aq, 89.75.Hc, 89.75.Fb

1. Introduction

1.1. *Background and research status*

The phase transitions exist in various physical processes. There are also have phase transitions phenomenon in the spread of disease.[2–6] In literatures,[7–10] the threshold of disease spread is analyzed by the quenched mean-field theory, discrete time Markov chain approach and N-intertwined approach. In literatures,[11–13] the prevalence threshold of the disease was increased and disease propagation was suppressed by increasing the clustering coefficient of the network. Some new literatures have found some effective propagation thresholds in networks through

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some computational methods such as edge compartment theory. Later, some researchers have studied behavioral communication based on the Watts threshold model[14] and founded that the phase transitions occurred in coevolutionary transmission. Literatures[1,15–19] study the coevolutionary spread of information and behavior, revealing the social impact mechanism and phase transition phenomenon of individual adoption behavior. In this paper, researchers also verify the phenomenon of phase transition in the process of information dissemination. The information dissemination is also affected by other factors in social networks, the most obvious factor is social influence. In the literatures,[20–26] they studied the impact of reputation on network communication, and studied the method of evaluating node credibility. The literatures[27–33] studied the impact of node influence on the information propagation, and proposed a method to maximize the nodes influence. The literatures[34–37] studied the propagation process of information in hierarchical networks, and found that the hierarchical networks have the function of promoting or suppressing to information dissemination. In the real world, the information dissemination is affected by social influence and social network structure. The literature[38] studied the higher-order network through the interpersonal relationship in reality. For the network of structure, literatures[39,40] provides a new way to construct the network. The literatures[41–47] study the role of social influences on the information dissemination, and the literature[48] examines the impact of repeated exposures on social network information dissemination. The study of the sociological literature[49] founded that the fusion of friends' attributes constitutes an individual self-network and affected by the communication behavior, and affecting the information dissemination. The above literature has made a lot of contributions on the research of propagation factors and network topology, but they did not study the phase transition in the process of information dissemination. Therefore, this paper proposes a new network model and a more realistic propagation model, and derives two effective propagation rates, and proposes two-phase transition processes in information propagation by defining information phase.

1.2. *Motivations*

The above discussion means that network structure, node attributes, and social influence will affect the information transmission, but most of researchers are not considered these effects on information transmission, and they are not considered the phenomenon of phase transition in information transmission. Literature[50] mentioned that the synergistic cumulative effect has influence on the spread of disease, and the phenomenon of phase transition in disease spread is revealed under the influence of synergistic cumulative effect. But they are different between the spread of disease and the spread of information. Therefore, this study mainly focuses on the phase transition process of information transmission in the real network, by analyzing the node influence, the three relationships of nodes and the social influence.

1.3. *Our work and contributions*

The main contributions of this paper are as follows: (1) Constructing a high-order network model. (2) Establishing a social cumulative effect model. (3) Constructing a propagation model with temporary forgetting and permanent forgetting characteristics. The above content is studied in the second part. (4) The proposed two-stage transition process of information dissemination and simulation experiments. It interacts with the communication space and simulates the communication on the constructed network. The simulation results are consistent with the actual situation of information transmission.

2. Modeling and Analysis

2.1. *Modeling of high-order network*

In the real world, there are intimate relationship and distant relationship in friends circle. This paper defines three relationships of each node, explicit, subexplicit, and implicit relationship.

Definition 1. Let G represent a high-order network with three types of edges.

$$G = (V, E, W). \tag{1}$$

$V = \{v_1, v_2, v_3, \ldots, v_n\}$ represents the set of nodes of G. $E = \{e_1, e_2, e_3, \ldots, e_m\}$ is the set of edges of G and the number of E is $|E|$, where $E_0 = \{e_i \,|\, e_i \in E, W(e_i) = 0\}$ represents the set of explicit edges and the number of E_0 is $|E_0|$, $E_1 = \{e_i \,|\, e_i \in E, W(e_i) = 1\}$ represents the set of subexplicit edges and the number of E_1 is $|E_1|$, $E_2 = \{e_i \,|\, e_i \in E, W(e_i) = 2\}$ represents the set of implicit edges and the number of E_2 is $|E_2|$. W represents the function of marking an edge, and $W(e_i) \in \{0, 1, 2\}$ represent the edge marks of explicit edge, subexplicit edge and implicit edge, respectively. After our analysis, set $\frac{|E_0|}{|E|} = 0.1$, $\frac{|E_1|}{|E|} = 0.2$, and $\frac{|E_0|}{|E|} = 0.7$. The network $G_E = (V_0, E_0)$, $G_S = (V_1, E_1)$, and $G_I = (V_2, E_2)$, where $V_0 = \{v_i, v_j \,|\, (v_i, v_j) \in E_0\}$, $V_1 = \{v_i, v_j \,|\, (v_i, v_j) \in E_1\}$, and $V_2 = \{v_i, v_j \,|\, (v_i, v_j) \in E_2\}$. The high-order network schematic diagram is showed following Fig. 1.

According to Definition 1, the basic propagation probability (propagation probability to neighbors without external force) of each node for explicit neighbors, subexplicit neighbors, and implicit neighbors is, respectively $\alpha_i \in [0.35, 0.4)$, $\alpha_i \in [0.25, 0.35)$, and $\alpha_i \in [0.2, 0.25)$. When they engage in social activities, they only care about themselves importance relative to their neighbors. This paper only considers the local importance of nodes. For its explicit neighbors, subexplicit neighbors and implicit neighbors, the relative local importance of each node is, respectively, $\psi_i \in [0.4, 0.5)$, $\psi_i \in [0.3, 0.4)$, and $\psi_i \in [0.2, 0.3)$.

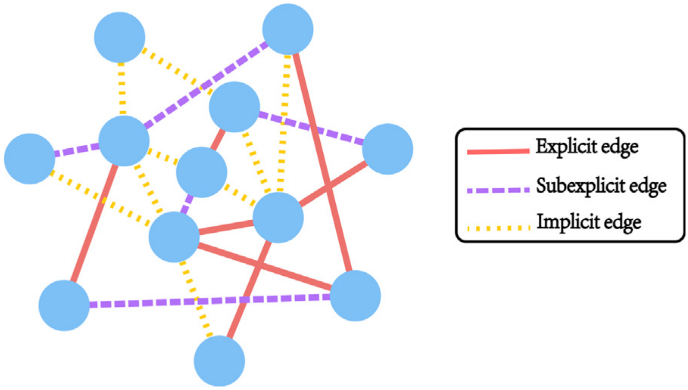

Fig. 1. (Color online) High-order network sketch map.

2.2. *Model of social cumulative effect*

In real life, we often participate in some voting activities, at the same time, we hope to get help from more people outside the circle of friends. This paper proposed the concept of accumulation and defined this phenomenon as the social cumulative effect (SCE). The SCE is defined as follows:

Definition 2. Let $\varphi_i(t)$ represents the SCE of node i at time t.

$$\varphi_i(t) = \sum_{k=0}^{2} \omega_k \Phi_i(t) = \Phi_i(t)\vec{\eta} \tag{2}$$

where $\vec{\eta} = (\omega_0, \omega_1, \omega_2)^T$, ω_i represents the corresponding network proportion in G. ω_0 represents the proportion of G_E, ω_1 represents the proportion of G_S, and ω_2 represents the proportion of G_I. $\Phi_i(t)$ represents the behavioral influence of node as follows:

Definition 3. Let $\Phi_i(t)$ is the behavioral influence of node.

$$\Phi_i(t) = f(\Omega_i(t)) = 1 - e^{-\Omega_i(t)} \tag{3}$$

where $\Omega_i(t) = \sum_{j\in\xi_i(t)} \psi_j \log_{10}(1 + \frac{d_{c\_j}}{d_{c\_\max}})$, $\Omega_i(t)$ represents the neighbor influence of node i, $\xi_i(t)$ represents the set of infected neighbors of node i at time t, ψ_j represents the importance of infected neighbor j relative to node i, d_{c_j} represents the degree of the infected neighbor j with the edge labeled $W \in \{0, 1, 2\}$, and $d_{c\_\max}$ represents the maximal degree of neighbors nodes of node i, these edges of neighbors nodes are marked as $W \in \{0, 1, 2\}$. The monotonicity and concavity of $f(\Omega_i(t))$ are consistent with recent studies,[51] which are about the changes of human behavior affected by social influence.

The individuals will be affected by the SCE, their propagation probability will be changed. The propagation probability of each node in the network is defined as follows:

Definition 4. Let α_u represent the dynamic propagation rate of node u.

$$\alpha_u = \begin{cases} \alpha_{u_0}\varphi_u(t)\ \widehat{(u,v)} = 0, & u \in V,\ v \in V, \\ \alpha_{u_1}\varphi_u(t)\ \widehat{(u,v)} = 1, & u \in V,\ v \in V, \\ \alpha_{u_2}\varphi_u(t)\ \widehat{(u,v)} = 2, & u \in V,\ v \in V. \end{cases} \tag{4}$$

Where α_{u_0} represents the basic propagation probability for explicit neighbors, α_{u_1} represents the basic propagation probability for subexplicit neighbors, and α_{u_2} represents the basic propagation probability for implicit neighbors.

2.3. *Modeling of propagation model*

Human memory has the characteristics of temporary forgetting and permanent forgetting. In this paper, temporary forgetting means that the forgotten information may be recalled by their neighbors, and permanent forgetting means that information is permanently forgotten over time. We constructed the Model 1 based on the forgetting characteristics (shown as in Fig. 2).

Here, S is the node that does not knows the message, I is the node that knows the message, $R1$ is the temporary forgetting nodes, and $R2$ is permanent-forgetting nodes.

When the information spreads throughout the network, and people stop talking about it, the information will be forgotten forever. We constructed the Model 2 based on the situation, its propagation model is shown in Fig. 3.

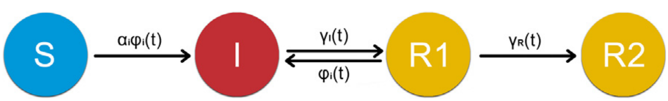

Fig. 2. (Color online) Model 1 (infected density prepeak model).

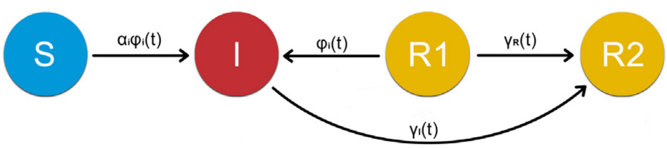

Fig. 3. (Color online) Model 2 (infected density postpeak model).

Definition 5. Let γ_I represents the forgetting probability of infected nodes at time t.

$$\gamma_I = \begin{cases} \gamma_{I,1} \ G_E \\ \gamma_{I,2} \ G_E \bigcup G_S \\ \gamma_{I,3} \ G_E \bigcup G_S \bigcup G_I \end{cases} \tag{5}$$

where $\gamma_{I,1}$, $\gamma_{I,2}$, and $\gamma_{I,3}$, respectively, indicate the forgetting probability of the node in the propagation network G_E, $G_E \bigcup G_S$, and $G_E \bigcup G_S \bigcup G_I$. Due to different intimacy relationships in different propagation networks, resulting in $\gamma_{I,1} \leq \gamma_{I,2} \leq \gamma_{I,3}$.

In the process of information dissemination, the interest of people for the information begin to decline after the most people known the information, the information will be forgotten forever.

Definition 6. Let γ_R represents the forgetting probability of nodes in the temporary forgetting state, thus γ_R defined as follows:

$$\gamma_R = \begin{cases} \gamma_{R,1} \ t \leq t_{i\_\max} \\ \gamma_{R,2} \ t > t_{i\_\max} \end{cases} \tag{6}$$

where $t_{i\_\max}$ represents the time when the infected density reaches its maximum. According to (2) and Model 1, two effective propagation rates of each node are obtained. The derivation process is as follows. From (2),

$$\begin{aligned} \varphi_k(t) &= \sum_{k=0}^{2} \omega_k f\left(\sum_{j\in\xi_i(t)} \psi_j \log_{10}\left(1+\frac{d_{c\_j}}{d_{c\_\max}}\right)\right) \\ &= \sum_{k=0}^{2} \omega_k \left(1-\exp\left(-\sum_{j\in\xi_i(t)} \psi_j \log_{10}\left(1+\frac{d_{c\_j}}{d_{c\_\max}}\right)\right)\right) \end{aligned} \tag{7}$$

where information is initially transmitted only in network G_E, $\vec{\eta} = (\omega_0, 0, 0)^T$, furthermore $\varphi_i(t) = \omega_0(1-e^{-\Omega_i(t)})$. Derivative of $\varphi_i(t)$

$$\varphi_i'(t) = (\omega_0(1-e^{-\Omega_i(t)}))' = \omega_0 \Omega_i'(t) e^{-\Omega_i(t)} \tag{8}$$

because of $e^{-\Omega_i(t)} > 0$, the growth of $\varphi_i'(t)$ is controlled by $\Omega_i'(t)$.

During the initial propagation, the information is propagated in the explicit network, nodes are affected by SCE from the explicit neighbors. In the experiment, randomly selected with ratio nodes as the initial infected nodes, these infected nodes have two distributions, called as IND 1 and IND 2.

IND 1. There is no neighbors nodes with state I of node i, that is $\xi_i(t) = \emptyset$.

For IND 1, $\Omega_i(t) = 0$, $\Omega_i'(t) = 0$, furthermore $\varphi_i(t) = 0$, $\varphi_i'(t) = 0$.

According to Definition 4, the transmission probability of infected node i is $\alpha_i(t) = \alpha_{i0}\varphi_i(t) = 0$, in other word, these nodes will not propagate information.

In real life, there is a phenomenon that individuals are deeply influenced by SCE. When a person knows some information, and his neighbors are not involved in discussing the information under the influence of noise, the information will not be distributed to any place.

IND 2. The number of neighbors nodes with state I of node i is s, that is $|\xi_i(t)| = s$.

Through IND 2, there is $\Omega_i(t) = \sum_{j\in\xi_i(t),|\xi_i(t)|=s} \psi_j \log_{10}(1 + \frac{d_{c\_j}}{d_{c\_\max}})$ for node i. From the Model 1:

$$|\xi_i(t)| = k_i\alpha_i(t-1) + \varphi_i(t-1)|R_{1i}| - \gamma_I(t-1)|I_i| \tag{9}$$

where $|\xi_i(t)|$ represents the number of elements in set $\xi_i(t)$, R_{1i} represents the set of neighbors whose state is R_1 in the neighbor of node i when the edge is marked as c, $|R_{1i}|$ represents the number of elements in set R_{1i}, I_i is also $\xi_i(t-1)$, and $|I_i|$ represents the number of elements in set I_i.

Let $|\xi_i(t)| = 0$,

$$k_i\alpha_i(t-1) + \varphi_i(t-1)|R_{1i}| = \gamma_I(t-1)|I_i| \tag{10}$$

$$\frac{\alpha_i(t-1)}{\gamma_I(t-1)}k_i + \frac{\varphi_i(t-1)}{\gamma_I(t-1)}|R_{1i}| = |I_i|\,. \tag{11}$$

Let $\lambda_{1,i} = \frac{\alpha_i(t-1)}{\gamma_I(t-1)}$, $\lambda_{2,i} = \frac{\varphi_i(t-1)}{\gamma_I(t-1)}$. Obviously, the change of $\Omega_i(t)$ is controlled by the two basic regeneration numbers $\lambda_{1,i}$ and $\lambda_{2,i}$. When $0 < \lambda_{1,i} < 1$, $0 < \lambda_{2,i} < 1$, there is $\Omega_i'(t) < 0$, and furthermore $\varphi_i'(t) < 0$. When $\lambda_{1,i} > 1$, $\lambda_{2,i} > 1$, there is $\Omega_i'(t) > 0$, and furthermore $\varphi_i'(t) > 0$. It can be seen that the Definition 2 is in line with the reality. When the information is spread in network G_E and network G_S, the derivation process is the same as above.

In order to observe the changes of SCE from a global perspective, the following definitions are given.

Definition 7. Let $\varphi(t)$ represents the average social cumulative effect (ASCE)

$$\varphi(t) = \frac{1}{N}\sum_{i=1}^{N} \varphi_i(t)\,. \tag{12}$$

Let Θ_I represents the probability that a node is connected to the state of I

$$\Theta_I = \frac{\sum_k kP(k)\rho_{k,I}}{\sum_k kP(k)} \tag{13}$$

where $\rho_{k,I}$ indicates the infected density of nodes with degree k and state I in the network G, and $P(k)$ is the degree distribution of the network G.

The average probability that a node with degree K is connected to multiple nodes with state I is

$$\Theta_K = \sum_{k=1}^{K} \binom{K}{k} (1-\Theta_I)^{K-k}{\Theta_I}^k\,. \tag{14}$$

Let N_I represent all neighbors set of the node with state I, let $P(\varphi'(t) > 0) = \frac{|N_I|\Theta_{\langle K_I \rangle}}{N} \geq \varepsilon_\varphi$, where $\langle K_I \rangle$ represents the average of the set N_I, obviously $P(\varphi'(t) > 0)$ is affected by $\rho_{k,I}$. It is certain that $\exists\varepsilon > 0$, $0 < \delta < 1$, when $|\rho_I = \sum_k \rho_{k,I}| \geq \varepsilon$, and $P(\varphi'(t) > 0) = 1 - \delta$.

$\lambda_{1,i}$ and $\lambda_{2,i}$ are obtained of each node according to (11). In order to characterize the phase transition of infected density, the following definitions are given:

Definition 8. Let λ_1 represent the first average effective propagation rate and λ_2 represent the second average effective transmission rate.

$$\begin{aligned} \lambda_1 &= \frac{1}{N}\sum_s \lambda_{1,i}(t) \\ \lambda_2 &= \frac{1}{N}\sum_s \lambda_{2,i}(t)\,. \end{aligned} \tag{15}$$

According to mean field theory, the thresholds of λ_1 and λ_2 are solved.

$$\begin{cases} \dfrac{d\rho_k(t)}{dt} = -\rho_k(t) + \lambda_k^{(1)}(t)k[1-\rho_k(t) - R_{1,k}(t)]\Theta(\rho_k(t)) \\ \qquad + \lambda_k^{(2)}(t)kR_{1,k}(t)\Theta(\rho_k(t)) \\ \dfrac{dR_{1,k}(t)}{dt} = -R_{1,k}(t) + \dfrac{1}{\lambda_k^{(2)}(t)}k\rho_k(t)\Theta(R_{1,k}(t))\,. \end{cases} \tag{16}$$

By solving (16), the approximate critical thresholds of λ_1 and λ_2 are obtained.

$$\tilde{\lambda}_1^{(c)} = \frac{\langle k \rangle}{N\langle k^2 \rangle \sum_s 1 - R_{1,s}}, \quad \tilde{\lambda}_2^{(c)} = \frac{\langle k \rangle}{N\langle k^2 \rangle \sum_s R_{1,s}} \tag{17}$$

According to the Definition 1, the network G is divided into three different relational networks. The information will be transmitted in these relational networks. For the convenience of description, G_A represent the propagation network at the certain moment. When the information propagates in different networks, $\tilde{\lambda}_1^{(c)}$ and $\tilde{\lambda}_2^{(c)}$ will be affected by the degree distribution of the network. For the convenience of description, the approximate propagation boundaries in spreading networks are as follows:

$$\tilde{\lambda}_{1_A}^{(c)} = \frac{\langle k_A \rangle}{N_A\langle k_A^2 \rangle \sum_s 1 - R_{1,s}}, \quad \tilde{\lambda}_{2_A}^{(c)} = \frac{\langle k_A \rangle}{N_A\langle k_A^2 \rangle \sum_s R_{1,s}} \tag{18}$$

where $\langle k_A \rangle$ is the average degree of network G_A, N_A is the number of nodes in the network G_A, and $\langle k_A^2 \rangle = \sum_k k^2 P_A(k)$, $P_A(k)$ is the degree distribution of network G_A.

3. Two Phase Transition Processes of Information Propagation

3.1. *Phase transition process based on infected density*

In real life, people give priority to spreading information to their closest friends. When information is transmitted to friends with different intimacy, it means that the scope and the influence of propagation information have changed, that is, the information have the phenomenon of phase transition. For instance, the film "Nezha" was released in mainland China. Initially, there were not many people known it; until its fans recommend it to their close friends and spread some information to the social network after watching the movie, Finally, this film is well known to most people, and the number of people watching the movie was gradually increasing. Based on the above situation, the information phase defined as follows:

Definition 9. According to relationship intimacy, the information is first transmitted through the network G_E. When the information infection density $i(t)$ meet the conditions: $i(t) < c_1$, the information density state is phase P_1. When the information infection density $i(t)$ meet the conditions: $c_1 \leq i(t) < c_2$, the network G_S is activated, the information density state is phase P_2. When information infection density $i(t)$ meet the conditions: $i(t) \geq c_2$, the network G_I is activated, the information density state is phase P_3. Where c_1 is the first-density threshold, c_2 is the second-density threshold.

Definition 10. P_1 (the initial phase) represents the information is only transmitted from the node to its explicit neighbor. P_2 (the growth phase) represents the information is transmitted from the node to its explicit neighbor and subexplicit neighbor. P_3 (the active phase) represents the information is propagated by the node to all its neighbors.

Proposition 1. *In network G_E, when the information infection density $i(t)$ meet $i(t) < c_1$ after the information begins to spread from the source node, the information is only transmitted in network G_E. When the information infection density $i(t)$ meet the conditions: $c_1 \leq i(t) < c_2$, the information propagated in network G_E and network G_S. When the information infection density $i(t)$ meet the conditions: $i(t) \geq c_2$, the information is transmitted in G_E, G_S, and G_I.*

According to Definition 10, the definition of $M(t)$ is given below.

Definition 11. $M(t)$ represents the phase transition function based on the change of density in the process of information transmission.

$$M(t) = i(t)\,. \tag{19}$$

The propagation diagram is shown in Fig. 4.

The red edge in Fig. 4 represents the active explicit edge. When the phase transition function $M(t)$ satisfies condition $c_1 \leq M(t) < c_2$, the first-order phase transition occurs in network G, that is, the phase P_1 changes into phase P_2. The green edge in the graph represents the activated subexplicit edge. When the phase transition function $M(t)$ satisfies the condition $c_2 \leq M(t)$, the second-order phase transition occurs in network G, that is, phase P_2 changes into phase P_3. The blue edge in the graph represents the activated recessive edge.

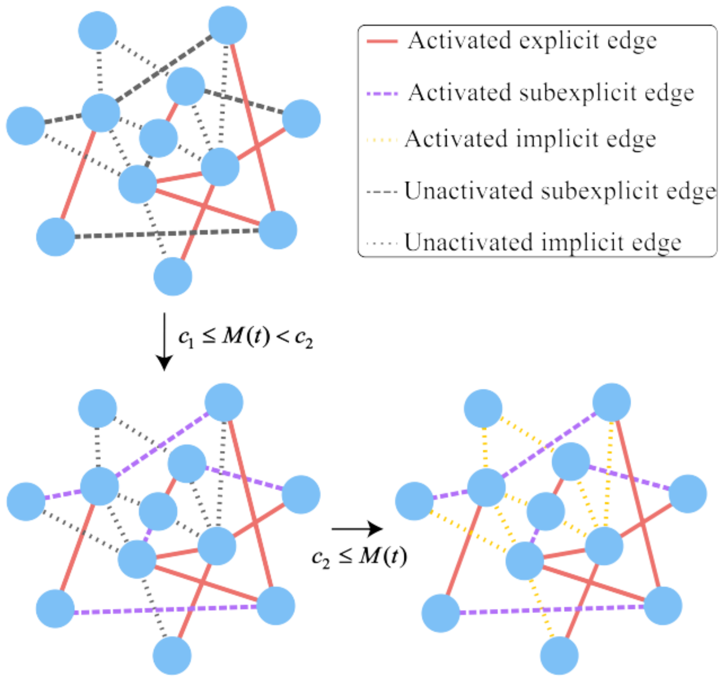

Fig. 4. (Color online) Diagram of phase transition process based on infected density.

According to the Model 1, the dynamic equation to information transmission in spreading network G_A is as follows:

$$\begin{cases} \dfrac{ds_A(t)}{dt} = -\dfrac{1}{N_A}\sum \alpha_{i_A}\varphi_{i_A}(t)I_{i_A}(t) \\ \dfrac{di_A(t)}{dt} = \dfrac{1}{N_A}\sum \alpha_{i_A}\varphi_{i_A}(t)I_{i_A}(t) + \dfrac{1}{N_A}\sum \varphi_{i_A}(t)R_{1i_A}(t) - \gamma_{I_A}(t)i_A(t) \\ \dfrac{dr_{1_A}(t)}{dt} = -\dfrac{1}{N_A}\sum \varphi_{i_A}(t)R_{1i_A}(t) - \gamma_{R_A}(t)r_{1_A}(t) \\ \dfrac{dr_{2_A}(t)}{dt} = \gamma_{R_A}(t)r_{1_A}(t) + \gamma_{I_A}(t)i_A(t). \end{cases} \tag{20}$$

For Eq. (20), in network G_A, the left side of the first equation represents the change speed of node density in the state S, with the right side of the equation indicate that the proportion of the population in the state S turns into the population in the state I. The left side of the second equation represents the change speed of the node density of state I, and the second term on the right side represents the proportion of the population whose state is $R1$ turning into that of state I. The third item represents the proportion of people whose state is I turning into those whose state is $R1$. The left side of the third equation represents the change speed of node density in the state of $R1$, and the third term on the right side represents the proportion of the population in the state of $R1$ turning into the population in the state of $R2$. The left side of the fourth equation represents the speed at which the density of nodes in the state $R2$ changes.

Where, in network G_A, N_A represents the number of nodes, α_{i_A} represents the propagation probability of node i, $\varphi_{i_A}(t)$ represents the SCE of node i, $I_{i_A}(t)$ represents the number of neighbors of node i whose state is I, $R_{1i_A}(t)$ represents the number of neighbors of node i whose state is $R1$, $\gamma_{I_A}(t)$ represents temporary forgetting probability, $i_A(t)$ represents the density of nodes with state I, and $\gamma_{R_A}(t)$ represents permanent forgetting probability. The specific expressions for $\gamma_{I_A}(t)$ and $\gamma_{R_A}(t)$ will be given later. $r_{1_A}(t)$ represents the density of nodes with state $R1$.

When $t = t_0$, $i_E(0, t_0)$ is the infected density in G_E at that time. From (20), let $G_A = G_E$,

$$i_E(0,t_0) = \int_0^{t_0} \frac{1}{N_E}\left[\sum \alpha_{i_E}\varphi_{i_E}(t)I_{i_E}(t) + \sum \varphi_{i_E}(t)R_{1i_E}(t)\right]dt - \int_0^{t_0} \gamma_{I_E}(t)i_E(t)dt \tag{21}$$

when $t = t_1$, $i_S(0, t_1)$ is the infected density in G_S at that time. From (20), let $G_A = G_S$,

$$i_S(0,t_1) = \int_0^{t_1} \frac{1}{N_S}\left[\sum \alpha_{i_S}\varphi_{i_S}(t)I_{i_S}(t) + \sum \varphi_{i_S}(t)R_{1i_S}(t)\right]dt - \int_0^{t_1} \gamma_{I_S}(t)i_S(t)dt. \tag{22}$$

When $t = t_2$, $i_I(0, t_2)$ is the infected density in G_I at that time. From (20), let $G_A = G_I$,

$$i_I(0,t_2) = \int_0^{t_2} \frac{1}{N_I}\sum \alpha_{i_I}\varphi_{i_I}(t)I_{i_I}(t) + \frac{1}{N_I}\sum \varphi_{i_I}(t)R_{1i_I}(t)dt - \int_0^{t_2} \gamma_{I_I}(t)i_I(t)dt. \tag{23}$$

Where N_E represents the number of nodes in network G_E, N_S represents the number of nodes in network G_S, and N_I represents the number of nodes in network G_I.

From Proposition 1, at time t,

$$G_A = \begin{cases} G_E, & c_1 > M(t) \\ G_E \bigcup G_S, & c_1 \le M(t) < c_2 \\ G_E \bigcup G_S \bigcup G_I, & c_2 \le M(t)\,. \end{cases} \tag{24}$$

When the information is propagated in G_E, If $\lambda_1 < \tilde{\lambda}^{(c)}_{1_E}$ and $\lambda_2 < \tilde{\lambda}^{(c)}_{2_E}$, the infection density increased slowly, and $M(t) < c_1$. If $\lambda_1 > \tilde{\lambda}^{(c)}_{1_E}$ and $\lambda_2 > \tilde{\lambda}^{(c)}_{2_E}$, $i_E(0,t)$ begins to grow. At time τ_1, the network G_S is activated, and $c_1 \le M(\tau_1) < c_2$. Furthermore, if $\lambda_1 > \tilde{\lambda}^{(c)}_{1_{E+S}}$ and $\lambda_2 > \tilde{\lambda}^{(c)}_{2_{E+S}}$, $i_E(0,t)$ and $i_S(\tau_1,t)$ proliferate. At time τ_2, the network G_I is activated, and $M(\tau_2) \ge c_2$. Therefore, when $M(t)$ satisfies the corresponding conditions, the density at time t is shown as follows:

$$i(t) = \begin{cases} \frac{1}{N}|\{v \,|\, v \in V_0, \text{ the state of vis I}\}|, & c_1 > M(t) \\ \frac{1}{N}\left|\{v \,|\, v \in V_0 \bigcup V_1, \text{the state of vis I}\}\right|, & c_1 \le M(t) < c_2 \\ \frac{1}{N}\left|\{v \,|\, v \in V_0 \bigcup V_1 \bigcup V_2, \text{the state of vis I}\}\right|, & c_2 \le M(t) \end{cases} \quad (25)$$

where $\tilde{\lambda}^{(c)}_{1_E}$ and $\tilde{\lambda}^{(c)}_{2_E}$ represent, respectively, the approximate propagation threshold of λ_1 and λ_2 in the network G_E, $\tilde{\lambda}^{(c)}_{1_{E+S}}$, and $\tilde{\lambda}^{(c)}_{2_{E+S}}$ represent, respectively, the approximate propagation threshold of λ_1 and λ_2 in the network $G_E \bigcup G_S$, τ_1 is the moment when $M(t)$ satisfies the condition $c_1 \le M(t) < c_2$, and τ_2 is the moment when $M(t)$ satisfies the condition $c_2 \le M(t)$. From (25), the phase transition of the information infection density is evident. Every time the space of transmission expands, the corresponding density of information infection increases in stages.

Establishing a high-order scale-free network with node number $N = 20000$ and average degree $\langle k \rangle = 13$. Random selection of initial infection density is 0.08, with all explicit edges in the network set to the active state. Setting $c_1 = 0.15$, $c_2 = 0.4$, $\gamma_{I,1} = 0.05$, $\gamma_{I,2} = 0.1$, $\gamma_{I,3} = 0.15$, $\gamma_{R,1} = 0.05$, and $\gamma_{R,2} = 0.15$. If $G_A = G_E$, $\vec{\eta} = (\omega_0, 0, 0)^T = (1, 0, 0)^T$. If $G_A = G_E \bigcup G_S$, $\vec{\eta} = (\omega_0, \omega_1, 0)^T = (0.3, 0.7, 0)^T$. If $G_A = G_E \bigcup G_S \bigcup G_I$, $\vec{\eta} = (\omega_0, \omega_1, \omega_2)^T = (0.1, 0.2, 0.7)^T$. When the phase transition function reaches the corresponding phase transition threshold value, the corresponding edges will be activated, so that the nodes can transmit information to the neighbors connected by the active edges. The experimental results are as follows.

In Fig. 5, the black curve is based on statistics of terrorist attacks in Sri Lanka through the statistics of the microhotspots system. The system is provided by Sina Company. The statistical time is from 0:00 to 20:00 on April 21, 2019, and from 0:00 to 05:00 on April 22, 2019.

From Fig. 5, the infected density begins to decrease at the initial stage of information transmission. The information is in the phase P_1. From Fig. 8, the growth rate of λ_1 and λ_2 is slow in the time interval [0, 50], that is, $\lambda_1'(t) \to 0$ and $\lambda_2'(t) \to 0$. From IND 1 and IND 2, it can be seen that in the early stage of information dissemination, node i conforming to IND 1 will change state from I to $R1$ through forgetting probability. For the node i conforming to IND 2, the information will be transmitted at the speed $\lambda_{1,i}$. Because $\lambda_1 > \tilde{\lambda}^{(c)}_{1_E}$ and $\lambda_2 > \tilde{\lambda}^{(c)}_{2_E}$ in network G_E (as shown in Fig. 7 in the red dot section and the purple dot section), it can be seen that the infected density increased at the time 50 in Fig. 5.

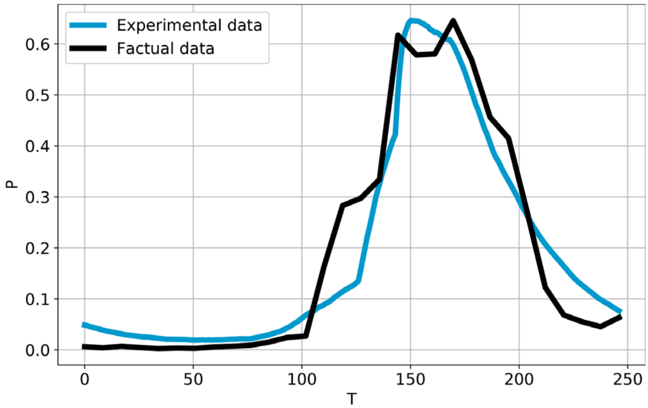

Fig. 5. (Color online) Experimental data and factual data.

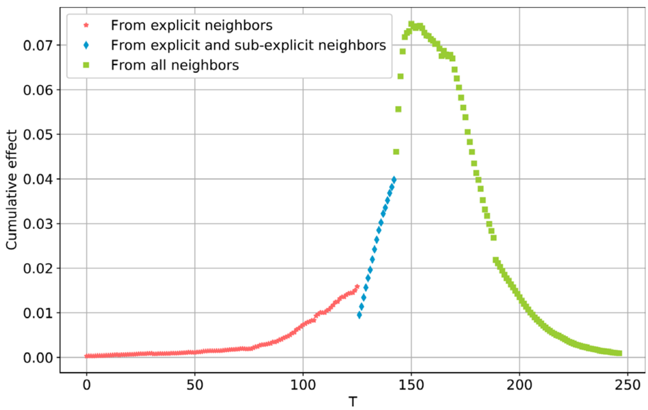

Fig. 6. (Color online) Average social cumulative effect.

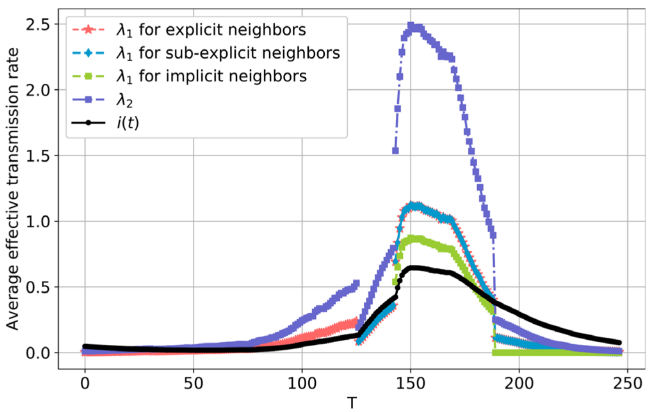

Fig. 7. (Color online) The changes of λ_1 and λ_2.

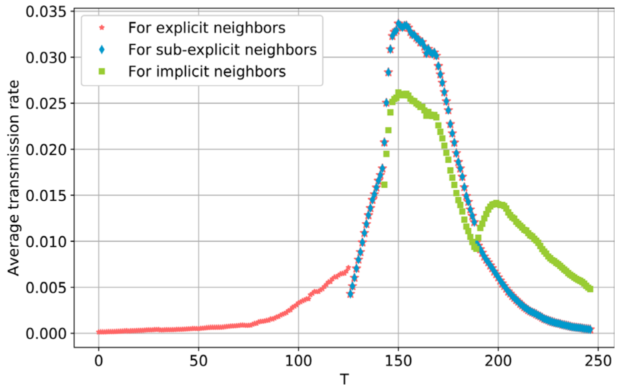

Fig. 8. (Color online) Average transmission rate for different typical neighbors.

When $c_1 \leq M(t) < c_2$, the information beginning to spread in network $G_E \bigcup G_S$ with activated network G_S. At this moment, there are $\lambda_1 \gg \tilde{\lambda}^{(c)}_{1_{E+S}}$ and $\lambda_2 \gg \tilde{\lambda}^{(c)}_{2_{E+S}}$, the information is in phase P_2, and the infected density increases rapidly, as shown the blue curve section in Fig. 5. The change of λ_1 and λ_2 are shown the red dot section in Fig. 7, the blue dot section, and the purple dot section. When $c_2 \leq M(t)$, the network G_I is activated, and the information is phase P_3. Information begins to spread in the network $G(G_E \bigcup G_S \bigcup G_I)$. Similarly, due to $\lambda_1 \gg \tilde{\lambda}^{(c)}_{1_G}$ and $\lambda_2 \gg \tilde{\lambda}^{(c)}_{2_G}$, the density increases dramatically, as shown the green curve section in Fig. 5.

In Figs. 5, 6, and 7, when the propagation network transforms from G_E to $G_E \bigcup G_S$ at time 126, the node radix and the forgetting probability increased, so, when the information spread to the second stage, the fracture appeared.

In Fig. 7, for implicit neighbors, $\lambda_1 = 0$ appears at time 187, in Fig. 8, the green curve shows an upward trend at that time. The algorithm of network planning allows the presence of nodes without implicit neighbors. Therefore, some nodes have not implicit neighbors in the set of infected nodes at time 187, this phenomenon appeared in Fig. 7. However, in Fig. 8, in calculating the average propagation probability to the implicit neighbors, the phenomenon is caused by the reduction of node cardinality, with those nodes existing without implicit neighbors.

In order to measure the similarity between the change trend of experimental data and that of the real case, the following definitions are given:

Definition 12. Let $L(x, y)$ represent the similarity between curve l_1 and curve l_2.

$$L(x, y) = \frac{x}{y}, \quad x \leq y \tag{26}$$

where x and y represent the area of curve l_1 and curve l_2.

According to Definition 12, the similarity between experimental data and real data is calculated, with the result being 78%, showing that the experimental results are consistent with the reality.

3.2. *Phase transition process based on information coupling*

In real life, people have different degrees of interest in each piece of information about the subject that is spread on the network. When an individual knows multiple information of the subject, the multiple information will be coupled at the individual, and the individual's degree of interest in the subject's-related information will be increased. At the same time, for different degrees of interest, individuals will choose different objects to spread based on their intimacy. The information entropy is used to measure an individual's degree of interest to information, and three types of information about the subject is set to rare information set $\wp_2$, relative rare information set $\wp_1$, and general information $\wp_0$. Each message has a different probability $P(A_j)$ of occurrence. In order to measure the node's total interest in the subject, the node entropy is defined as shown

Definition 13. Let H_i represents the node entropy of node i, with the definition as follows:

$$H_i = \sum_j \log_{10}\left(\frac{1}{P(A_j)}\right). \tag{27}$$

Remark 1. Generally, the smaller the probability of an event, the more interest people have in the event, and the greater the information entropy of the event. When $a_{i1} \le H_i < a_{i2}$, node i tends to share the event information to the explicit and subexplicit neighbors. When $a_{i2} \le H_i$, node i tends to share the event information with all neighbors. Where a_{i1} is the first information entropy threshold, a_{i2} is the second information entropy threshold.

Proposition 2. *On the basis of Remark* 1, *if* $H_i < a_{i1}$, *the information is shared only with the explicit neighbor of node* i; *if* $a_{i1} \le H_i < a_{i2}$, *the information is shared only with explicit neighbor and subexplicit neighbor of node* i.

The higher the individual's interest in information, the higher the motivation for sharing. Therefore, the node's information sharing probability function is defined as shown follows.

Definition 14. Let $\zeta(x)$ represent the probability function of information sharing.

$$\zeta(x) = e^{-\theta x} \tag{28}$$

where θ is the sharing coefficient.

According to Proposition 2, when the node i receives n pieces of information about a topic, and if these information are general information. When the node entropy meets the condition $H_i < a_{i1}$, the phase transition of node i will not occur, and it only will communicate with the explicit friends. When there are m_1 relative rare information in n pieces of information, the node entropy meets the condition $a_{i1} \le H_i < a_{i2}$. With the increasement of the number of infected neighbor nodes of node i, the node i will produce the first-order phase transition, and the node will spread information to the explicit neighbors and subexplicit neighbors. When there are m_2 rare information in n pieces of messages, the number of infected neighbor nodes will increase, the node entropy meets the condition $a_{i2} \leq H_i$, which will produce the second-order phase transition, and the node will spread the information to its all neighbors.

Definition 15. According to Proposition 2, let $M_i(t)$ represent node i's phase transition function based on information coupling.

$$M_i(t) = \frac{|I_i|}{d_i} H_i \,. \tag{29}$$

According to Proposition 2, the phase transition conditions at node i are set to be c_{i1} and c_{i2}, which are expressed as follows:

$$\begin{aligned} a_{i1} &= \frac{C_{n_1}^{m_1}}{\sum_{i=1}^{|\wp_1|} C_{|\wp_1|}^{i}} \prod_{j \in D_i^{(1)}, |D_i^{(1)}| = m_1} P(A_j) \Bigg|_{D_i^{(1)} \subseteq \wp_1}, \quad c_{i1} = \frac{1}{2} \log_{10} \left(\frac{1}{a_{i1}} \right), \\ a_{i2} &= \frac{C_{n_2}^{m_2}}{\sum_{i=1}^{|\wp_2|} C_{|\wp_2|}^{i}} \prod_{j \in D_i^{(2)}, |D_i^{(2)}| = m_2} P(A_j) \Bigg|_{D_i^{(2)} \subseteq \wp_2}, \quad c_{i2} = \frac{1}{2} \log_{10} \left(\frac{1}{a_{i2}} \right). \end{aligned} \tag{30}$$

According to Definition 9, if $M_i(t) < c_{i1}$, the information phase is phase P_1 at node i; if $c_{i1} \leq M_i(t) < c_{i2}$, the information phase is phase P_2 at node i; if $M_i(t) \geq c_{i2}$, the information phase is phase P_2.

Table 1. Parameter description.

Parameter	Definition
$D_i^{(1)}$	Relatively rare information set received by node i
$\|D_i^{(1)}\|$	Number of elements in set $D_i^{(1)}$
$D_i^{(2)}$	Rare information set received by node i
$\|D_i^{(2)}\|$	Number of elements in set $D_i^{(2)}$
$\wp_1$	On the set of all relatively rare information about the subject
$\|\wp_1\|$	Number of elements in set $\wp_1$
$\wp_2$	On the set of all rare information about the subject
$\|\wp_2\|$	Number of elements in set $\wp_2$
a_{i1}	Probability of a_{i1} receiving m_1 messages from $\wp_1$ as node i
a_{i2}	Probability of a_{i2} receiving m_2 messages from $\wp_2$ as node i
c_{i1}	The number of neighbors directly infected by node i is half of the number of neighbors and the expected entropy of node receiving m_1 relatively rare messages.
c_{i2}	The number of neighbors directly infected by node i is half of the number of neighbors and the expected entropy of node receiving m_2 rare messages

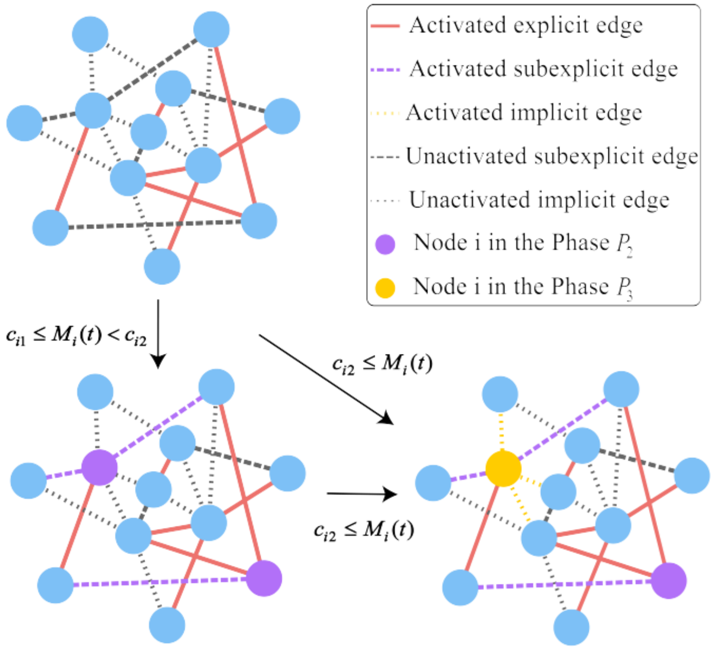

Fig. 9. (Color online) Diagram of phase transition diagram based on information coupling.

The propagation diagram is shown in Fig. 9.

When the phase transition function $M_i(t)$ of node i satisfies the corresponding conditions, the corresponding edges are activated, as shown below.

$$E_i = \begin{cases} E_0, & c_{i1} > M_i(t) \\ E_0 \bigcup \left(\bigcup\limits_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1\}} (v_i, v_j) \right), & c_{i1} \le M_i(t) < c_{i2} \\ E_0 \bigcup \left(\bigcup\limits_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1,2\}} (v_i, v_j) \right), & c_{i2} \le M_i(t) \end{cases} \tag{31}$$

when $c_{i1} > M_i(t)$, $G_A = G_E$, and the information phase is phase P_1 on network G. When

$$c_{i1} \le M_i(t) < c_{i2}, \quad G_A = \left(V_0 \bigcup I_G, E_0 \bigcup \left(\bigcup\limits_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1\}} (v_i, v_j) \right) \right),$$

and the information phase is phase P_2 on network G. When

$$c_{i2} \le M_i(t), \quad G_A = \left(V_0 \bigcup I_G, E_0 \bigcup \left(\bigcup\limits_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1,2\}} (v_i, v_j) \right) \right),$$

and the information phase is phase P_3 on network G. Where $I_G = \{v|v \in V, \textit{the state of vis } I\}$.

When the propagation network is expanded, some nodes in the network know at least event $A_i \in \wp_2$ or $A_j \in \wp_1$. For event A_i or A_j, its expansion speed in network G_A is as follows:

$$\frac{di_A^{(A_i)}(t)}{dt} = \zeta(P(A_i))\langle k_A\rangle s_A(t) i_A^{(A_i)}(t) + \varphi_A(t) r_{1_A}(t) - \gamma_I(t) i_A^{(A_i)}(t) \tag{32}$$

where $i_A^{(A_i)}(t)$ indicates the density of the node that knows the event A_i in the network G_A.

Let $\frac{di_A(t)}{dt} = 0$, the effective expansion speed of A_i is $\lambda_{A_i} \approx \frac{\zeta(P(A_i))+\varphi_A(t)}{\gamma_I(t)}$, and at time t, there must be $\lambda_{A_i} \geq 1$. Therefore, for event A_i, it will spread to the network at the effective expansion speed λ_{A_i}, expanding the information dissemination channel for the relevant information. There are two ways of expansion.

(1) Twice phase transitions occur in the propagation space at node i
(2) Once phase transition in the propagation space at node i

For (1), in the network, event A_i about the subject occurs at time t_1, and the node received the information activates the subexplicit connection with it. Event A_j about the subject occurs at time t_2, and the node receiving the information activating all the sides connected with it. where $t_1 < t_2$.

For (2), at least one rare event A_j about the subject occurs at time t_1 in the network, with the node receiving the information activating all the edges connected with it.

A high-order scale-free network is established with node number $N = 20000$ and average degree $\langle k\rangle = 13$. Random selection of initial infection density is 0.08, and all explicit edges in the network are set to active state. Setting $c_1 = 0.15$, $c_2 = 0.4$, $\gamma_{I,1} = 0.05$, $\gamma_{I,2} = 0.1$, $\gamma_{I,3} = 0.15$, $\gamma_{R,1} = 0.05$, and $\gamma_{R,2} = 0.15$. If $G_A = G_E$, then $\vec{\eta} = (\omega_0, 0, 0)^T = (1, 0, 0)^T$. If $G_A = (V_0 \bigcup I_G, E_0 \bigcup (\bigcup_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1\}} (v_i, v_j)))$, then $\vec{\eta} = (\omega_0, \omega_1, 0)^T = (0.3, 0.7, 0)^T$. If $G_A = (V_0 \bigcup I_G, E_0 \bigcup (\bigcup_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1,2\}} (v_i, v_j)))$, then $\vec{\eta} = (\omega_0, \omega_1, \omega_2)^T = (0.1, 0.2, 0.7)^T$. Let $\wp_0 = \{A_1, A_2, A_3, A_4\}$, $\wp_1 = \{A_5, A_6, A_7\}$, and $\wp_2 = \{A_8, A_9, A_{10}\}$. Let $m_1 = 1$ and $m_2 = 1$.

Let

$$\begin{cases} P(e) \in [0.75 - 0.9], & e \in \wp^0 \\ P(e) \in [0.4 - 0.6], & e \in \wp^1 \\ P(e) \in [0.005 - 0.01], & e \in \wp^2 \end{cases} \tag{33}$$

For (1), the experimental results are as follows:

In Fig. 10, the real data are obtained from the micro hotspot system. This system is provided by Sina company. The statistical time is from 0:00 to 20:00 on

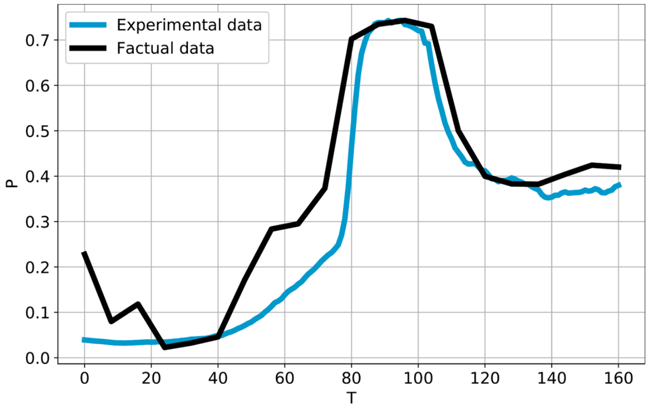

Fig. 10. (Color online) Experimental data and factual data.

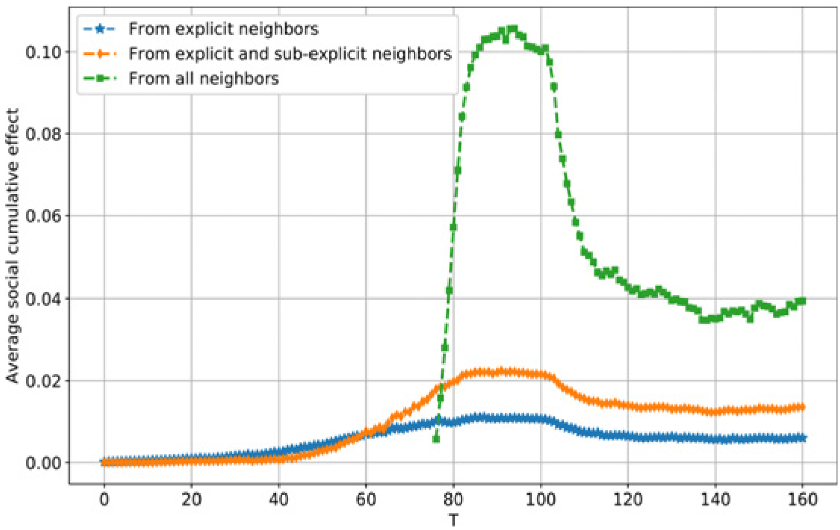

Fig. 11. (Color online) Average social cumulative effect.

October 1, 2019. The place of occurrence is in China. The search keyword is Air flag guard echelon open the prologue of the parade.

In Fig. 10, early in the spread of information, the density begins to decrease, and the density begins to rise again at time 40, and the information phase is phase P_1 on the network G. At time 40, the orange curve starts to grow rapidly in Fig. 11. Due to the occurrence of a relatively rare message about the subject in the network, the node receiving the message activates the connection of its subexplicit neighbor, that is $c_{i1} \leq M_i(t) < c_{i2}$, and the information phase is phase P_2 on the network G. At the same time, due to the expansion of propagation space in the network G_A and the increase of the propagation motivation of the nodes, λ_2 and λ_1 of explicit

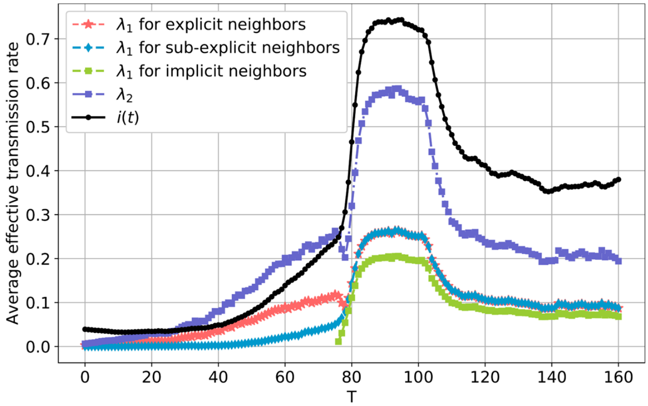

Fig. 12. (Color online) The changes of λ_1 and λ_2.

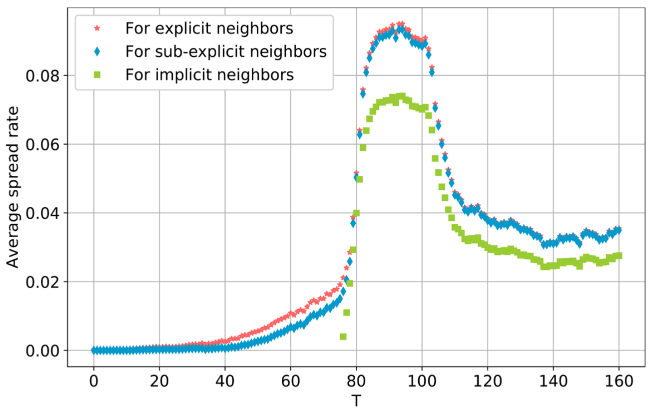

Fig. 13. (Color online) Average transmission rate for different typical neighbors.

neighbors and subexplicit neighbors grow rapidly, as shown in Fig. 12. Similarly, the average transmission probability of explicit and subexplicit neighbors is shown in Fig. 13. It also increases rapidly, which further leads to the rapid increase of infected density, as is evident in Fig. 10. At time 78, a rare message about the subject occurs in the network. When a rare message about the subject occurs, people based on understanding of the subject other information and strong curiosity about the information, it will be quickly transmitted to all the neighbors around it, that is. At the moment, the information phase is phase P_3 on the network G. Figure 11 shows that the green curve is growing exponentially. As a result of such a group behavior, the ASCE from all neighbors on each node in Fig. 11 increases exponentially, with the growth rates of λ_1 and λ_2 increasing rapidly. Further infection density increases sharply at time 78. According to the experiment, it was found that the phenomenon of information of synergistic transmission occurred during the phase transition.

In Fig. 12, λ_2 and λ_1 of explicit neighbors and sub-explicit neighbors decrease at time 78. At this moment, a rare message about the subject appears in the network, which propagates to the network at an exponential speed, so that the transmission network is instantaneously transformed from G_A to G. The base number of the network nodes increases sharply, which results in this phenomenon. Where $G_A = (V_0 \bigcup I_G, E_0 \bigcup (\bigcup_{v_i \in I_G, v_j \in V, W((v_i,v_j)) \in \{0,1\}} (v_i, v_j)))$, in Fig. 10, the process of decreasing infection density is compared with that of decreasing infection density in Fig. 4. In Fig. 10, the infection density $i(t) \to \varepsilon_I$ is seen clearly. In the network, some nodes knowing most of the subjects information lose interest for this type of information, they will forget it for a long time; while the nodes with interest in this kind of message are still discussing these messages with their surroundings and exchanging information that they do not know, so this phenomenon occurs, and the information is said to be in the retention phase (P_4).

According to Definition 12, the similarity between experimental data curve and real data curve is 85%, showing that the experimental results being consistent with the reality.

For (2), The experimental results are as follows:

In Fig. 14, the real data are obtained from the microhotspot system provided by Sina company. The statistical time is from 0:00 to 18:00 on April 22, 2019. The place of occurrence is in China. In the case base of the system, the search keyword is Tesla spontaneous combustion.

In Fig. 17, the average propagation probability of subexplicit neighbors and implicit neighbors changes at time 30. Similarly, in Fig. 16, there is λ_1 for subexplicit neighbors and implicit neighbors. At this time, at least one rare information about the subject occurs in the network, and the node receiving the information activates its subexplicit and implicit links and propagates the message to all neighbors. At this time, the transmission network is G. The information phase changes the phase P_1 into the phase P_3. Thus, when $\lambda_1 > \tilde{\lambda}^{(c)}_{1_G}$ and $\lambda_2 > \tilde{\lambda}^{(c)}_{2_G}$, the infected density

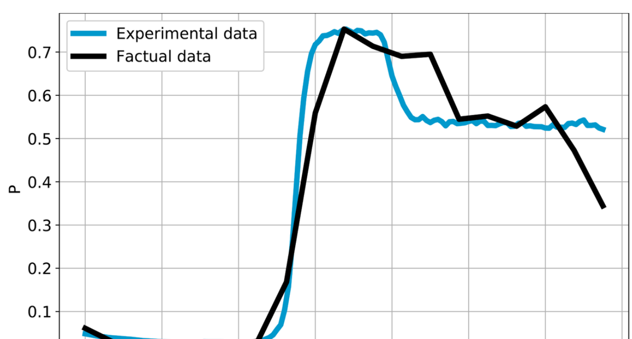

Fig. 14. (Color online) Experimental data and factual data.

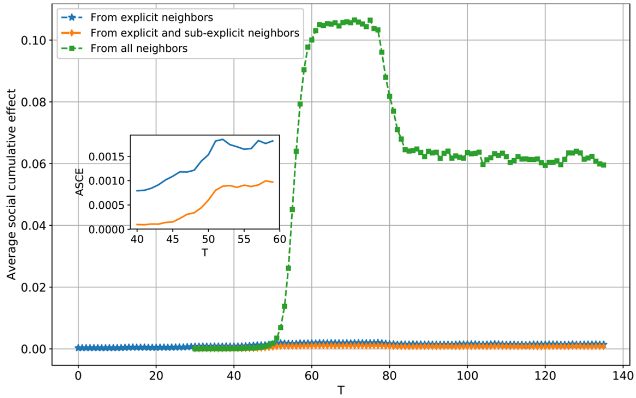

Fig. 15. (Color online) Average social cumulative effect.

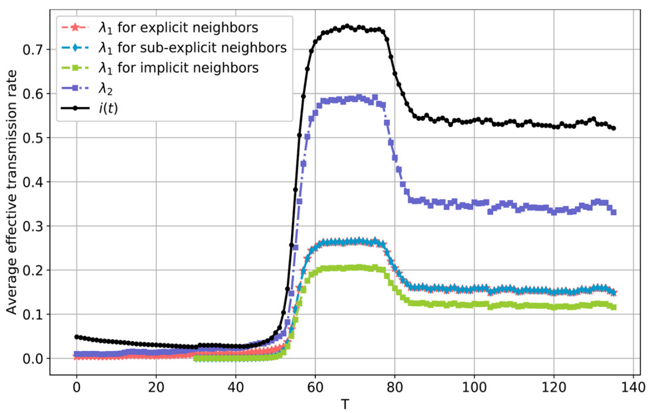

Fig. 16. (Color online) The changes of λ_1 and λ_2.

increases exponentially, as shown in Fig. 14. Therefore, λ_1 of different types of neighbors and λ_2 increased rapidly, as shown in Fig. 17. At this time, the information phase changes P_1 into P_3.

As shown in the subgraph in Fig. 16, the orange curve and the blue curve have changed slightly, while the green curve changes dramatically in the time interval [40, 60]. This is because the blue curve represents the ASCE of nodes that only activate explicit edges. The orange curve represents the ASCE of nodes that only activate explicit and subexplicit edges. However, when the rare message about the subject occurs and propagates in the network, it activates all the edges of most nodes quickly, so the number of nodes activating only the explicit edge or only the explicit edge and the subexplicit edge is minimal, with the curve changing slightly. According to Definition 12, the similarity between the experimental data curve and the real data curve is 94%, an indication that the experimental results are consistent with reality.

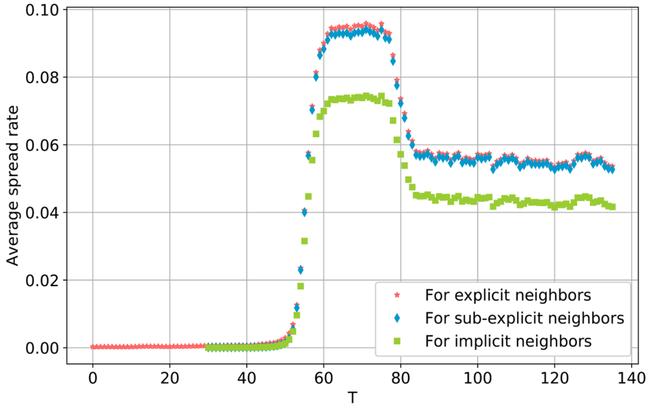

Fig. 17. (Color online) Average transmission rate for different typical neighbors.

4. Conclusion

This paper studies the information phase transition in higher-order networks. A new high-order network model is proposed based on node relationships. Improved the classic SIR model by analyzing the characteristic of personal oblivion, and proposed the two propagation models with the characteristics of permanent oblivion and temporary oblivion. By analyzing the social cumulative effect and SI2R propagation model, we found node has two effective propagation rates and the propagation ability of node is affected by the social cumulative effect. The phenomenon of information of synergistic transmission was found in the process of information phase transition of "information coupling." When the individual did not know information, the information will stay on the network and continue to spread, forming the phenomenon of information retention. Finally, it is found that the theory is in line with reality through comparison with real cases.

Acknowledgments

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