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Research Article

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Multiplicity of concentrating solutions for a class of magnetic Schrödinger-Poisson type equation

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Abstract: In this paper, we study the following nonlinear magnetic Schrödinger-Poisson type equation

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u + \epsilon^{-2}(|x|^{-1} \star |u|^2)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{C}), \end{cases}$$

where $\epsilon > 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ are continuous potentials. Under a local assumption on the potential V, by variational methods, penalization technique, and Ljusternick-Schnirelmann theory, we prove multiplicity and concentration properties of nontrivial solutions for $\varepsilon > 0$ small. In this problem, the function f is only continuous, which allow to consider larger classes of nonlinearities in the reaction.

Keywords: Schrödinger-Poisson system, Magnetic field, Multiple soutions, Variational methods

MSC: 35J60, 35J25

1 Introduction and main results

In this paper, we are concerned with multiplicity and concentration results for the following Schrödinger-Poisson type equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u + \epsilon^{-2}(|x|^{-1} \star |u|^2)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $u \in H^1(\mathbb{R}^3, \mathbb{C})$, $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $f \in C(\mathbb{R}, \mathbb{R})$, the magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ is Hölder continuous with exponent $\alpha \in (0, 1]$, and the convolution potential is defined by $|x|^{-1} \star |u|^2 = \int_{\mathbb{R}^3} |x - y|^{-1}|u(y)|^2 dy$.

Problem (1.1) arises in quantum mechanics, abelian gauge theories, plasma physics, and so on which can be used to simulate the mutual interactions of many particles. In fact, the linear Schrödinger equation describes the behavior of a single particle. However, the interaction among particles can be simulated by adding a nonlinear term f. Moreover, the convolution potential is a solution of Poisson equation which implies that the particles move in their own gravitational field generated by the probability density of particles via classical Newton field equation. Therefore, problem (1.1) can be regarded as the coupling of the Schrödinger equation and Poisson equation.

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There is a vast literature concerning the existence and multiplicity of solutions for nonlinear equation without magnetic field. We notice that Fiscella, Pucci and Zhang [16] studied the existence of solutions for p-fractional Hardy-Schrödinger-Kirchhoff systems with critical nonlinearities, Ji and Radulescu [17] considered the multiplicity of multi-bump solutions for quasilinear elliptic equations with variable exponents and critical growth in $\mathbb{R}^N$, for more results, we refer to the Monograph [25]. Recently, by using the method of Nehari manifold and Ljusternik-Schnirelmann theory, He [21] proved the multiplicity and concentration of solutions of problem (1.1) for $f \in C^1(\mathbb{R}, \mathbb{R})$ and the potential satisfying a global condition introduced by Rabinowitz [26]. In [22], on the similar assumptions, He and Zou studied the existence and concentration behavior of ground state solutions for a class of Schrödinger-Poisson system with critical the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$. Then, under a local assumption introduced by del Pino and Felmer [14], He and Zou [23] studied the multiplicity of concentrating positive solutions for Schrödinger-Poisson equations with critical nonlinear $f \in C^1(\mathbb{R}, \mathbb{R})$. For further results about existence and nonexistence of solutions, multiplicity of solutions, ground states, semi-classical limit and concentrations of solutions for Schrödinger-Poisson system(see [1–4, 11, 12, 27, 28, 31, 35] and the references therein).

On the other hand, the magnetic nonlinear Schrödinger equation (1.1) has been extensively investigated by many authors applying suitable variational and topological methods (see [5–7, 9, 10, 13, 15, 18–20, 32–34] and references therein). It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [15]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed. In [34], Xiang, Rădulescu and Zhang studied multiplicity and concentration of solutions for magnetic relativistic Schrödinger equations, Xia [32] studied a critical fractional Choquard-Kirchhoff problem with magnetic field. In particular, due to our scope, we want to mention [36] where the authors studied a Schrödinger-Poisson type equation with magnetic field by using the method of the Nehari manifold, the penalization method and Ljusternik-Schnirelmann category theory for subcritical nonlinearity $f \in C^1$. If f is only continuous, then the arguments in [36] failed.

In this paper, motivated by [23, 29, 36], for the case f is only continuous, we intend to prove multiplicity and concentration of nontrivial solutions for problem (1.1). We note that, due to the appearance of magnetic field $A(x)$, problem (1.1) will be more difficult in employing the methods and some estimates. On the other hand, due to the nonlocal term $|x|^{-1} \star |u|^2$, some estimates are also more complicated.

Throughout the paper, we make the following assumptions on the potential V:

$(V1)$There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$;

$(V2)$There exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, let the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying:

$(f1)$ $f(t) = 0$ if $t \leq 0$, and $\lim_{t \to 0^+} \frac{f(t)}{t} = 0$;

$(f2)$ there exists $q \in (4, 6)$ such that

$$\lim_{t \to +\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0;$$

$(f3)$ there is a positive constant $\theta > 4$ such that

$$0 < \frac{\theta}{2} F(t) \leq t f(t), \qquad \forall\, t > 0, \quad \text{where } F(t) = \int_0^t f(s)ds;$$

$(f4)$ $\frac{f(t)}{t}$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following:

Theorem 1.1. *Assume that V satisfies $(V1)$, $(V2)$ and f satisfies $(f1)$–$(f4)$. Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^3 : dist(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.1) *has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ε_n} one of these solutions of problem* (1.1) *for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^3$ the global maximum point of $|u_{\varepsilon_n}|$, then*

$$\lim_{\varepsilon_n \to 0^+} V(\eta_{\varepsilon_n}) = V_0.$$

The paper is organized as follows. In Section 2 we introduce the functional setting and give some preliminaries. In Section 3, we study the modified problem. We prove the Palais-Smale condition for the modified functional and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the autonomous problem associated. It allows us to show the modified problem has the multiple soutions. Finally, in Section 5, we give the proof of Thereom 1.1.

Notation

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line;
- $B_R(y)$ denotes the open disk centered at $y \in \mathbb{R}^3$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^3$;
- $\|\cdot\|$, $\|\cdot\|_q$, and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^3, \mathbb{R})$, $L^q(\mathbb{R}^3, \mathbb{R})$, and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^3$.

2 Abstract setting and preliminary results

In this section, we present the functional spaces and some useful preliminary remarks which will be useful for our arguments.

For $u : \mathbb{R}^3 \to \mathbb{C}$, let us denote by

$$\nabla_A u := \left(\frac{\nabla}{i} - A\right) u,$$

and

$$D_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in L^6(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R})\}.$$

and

$$H_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in D_A^1(\mathbb{R}^3, \mathbb{C}) : u \in L^2(\mathbb{R}^3, \mathbb{C}))\}.$$

The space $H_A^1(\mathbb{R}^3, \mathbb{C})$ is an Hilbert space endowed with the scalar product

$$\langle u, v\rangle := \mathrm{Re} \int_{\mathbb{R}^3} \left(\nabla_A u \overline{\nabla_A v} + u\bar{v}\right) dx, \quad \text{for any } u, v \in H_A^1(\mathbb{R}^3, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover we denote by $\|u\|_A$ the norm induced by this inner product.

On $H_A^1(\mathbb{R}^3, \mathbb{C})$ we will frequently use the following diamagnetic inequality (see e.g. [24, Theorem 7.21])

$$|\nabla_A u(x)| \geq |\nabla |u(x)||. \tag{2.1}$$

Moreover, making a simple change of variables, we can see that (1.1) is equivalent to

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u + (|x|^{-1} \star |u|^2)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

Let H_ε be the Hilbert space obtained as the closure of $C_c^\infty(\mathbb{R}^3, \mathbb{C})$ with respect to the scalar product

$$\langle u, v\rangle_\varepsilon := \mathrm{Re} \int_{\mathbb{R}^3} \left(\nabla_{A_\varepsilon} u \overline{\nabla_{A_\varepsilon} v} + V_\varepsilon(x) u\overline{v}\right) dx$$

and let us denote by $\|\cdot\|_\varepsilon$ the norm induced by this inner product.

The diamagnetic inequality (2.1) implies that, if $u \in H^1_{A_\varepsilon}(\mathbb{R}^3, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$ and $\|u\| \leq C\|u\|_\varepsilon$. Therefore, the embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^3, \mathbb{C})$ is continuous for $2 \leq r \leq 6$ and the embedding $H_\varepsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ is compact for $1 \leq r < 6$.

By using the continuous embedding $H^1(\mathbb{R}^3, \mathbb{R}) \hookrightarrow L^r(\mathbb{R}^3, \mathbb{R})$ for $2 \leq r \leq 6$, we can see that

$$H^1(\mathbb{R}^3, \mathbb{R}) \hookrightarrow L^{\frac{12}{5}}(\mathbb{R}^3, \mathbb{R}). \tag{2.3}$$

For any $u \in H_\varepsilon$, we get $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$, and the linear functional $\mathcal{L}_{|u|} : D^{1,2}(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ given by

$$\mathcal{L}_{|u|}(v) = \int_{\mathbb{R}^3} |u|^2 v dx$$

is well defined and continuous in view of the Hölder inequality and (2.4). Indeed, we can see that

$$\left|\mathcal{L}_{|u|}(v)\right| \leq \Big(\int_{\mathbb{R}^3} |u|^{\frac{12}{5}} dx\Big)^{\frac{5}{6}} \Big(\int_{\mathbb{R}^3} |v|^6 dx\Big)^{\frac{1}{6}} \leq C\|u\|^2_{D^{1,2}}\|v\|_{D^{1,2}}, \tag{2.4}$$

where

$$\|v\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} (|x|^{-1} \star |v|^2)|v|^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{-1}|u(x) - u(y)|^2 dxdy.$$

Then, by the Lax-Milgram Theorem, there exists a unique $\phi_{|u|} \in D^{1,2}(\mathbb{R}^3, \mathbb{R})$ such that

$$-\Delta\phi_{|u|} = |u|^2, \quad \text{in } \mathbb{R}^3.$$

Therefore we obtain the following t-Riesz formula

$$\phi_{|u|}(x) = c \int_{\mathbb{R}^3} |x-y|^{-1}|u(y)|^2 dy.$$

In the sequel, we will omit the constant for simplicity. The function $\phi_{|u|}$ possesses the following properties.

Lemma 2.1. *For any $u \in H_\varepsilon$, we have*
(i) $\phi_{|u|} : H^1(\mathbb{R}^3, \mathbb{R}) \to D^{1,2}(\mathbb{R}^3, \mathbb{R})$ *is continuous and maps bounded sets into bounded sets;*
(ii) *if $u_n \rightharpoonup u$ in H_ε, then $\phi_{|u_n|} \rightharpoonup \phi_{|u|}$ in $D^{1,2}(\mathbb{R}^3, \mathbb{R})$, and*

$$\liminf_n \int_{\mathbb{R}^3} \phi_{|u_n|^2}|u_n|^2 dx \leq \int_{\mathbb{R}^3} \phi_{|u|^2}|u|^2 dx;$$

(iii) $\phi_{|ru|} = r^2\phi_{|u|}$ *for all $r \in \mathbb{R}$ and $\phi_{|u(\cdot+y)|} = \phi_{|u|}(x+y)$;*
(iv) $\phi_{|u|} \geq 0$ *for all $u \in H_\varepsilon$ and we have*

$$\|\phi_{|u|}\|_{D^{1,2}} \leq C\|u\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)} \leq C\|u\|^2_\varepsilon, \ \ \textit{and} \int_{\mathbb{R}^3} \phi_{|u|}|u|^2 dx \leq C\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \leq C\|u\|^4_\varepsilon.$$

The proof of Lemma 2.1 is similar to one in [27, 35], so we omit it.

3 The modified problem

To study problem (1.1), or equivalently (2.2) by variational methods, we shall modify suitably the nonlinearity f so that, for $\varepsilon > 0$ small enough, the solutions of such modified problem are also solutions of the original one. More precisely, we choose $K > 2$. By $(f4)$ there exists a unique number $a > 0$ verifying $Kf(a) = V_0$, where V_0 is given in $(V1)$. Hence we consider the function

$$\tilde{f}(t) := \begin{cases} f(t), & t \le a, \\ V_0/K, & t > a. \end{cases}$$

Now we introduce the penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$

$$g(x,t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t), \tag{3.1}$$

where χ_Λ is the characteristic function on Λ and $G(x,t) := \int\limits_0^t g(x,s)ds$.

In view of $(f1)$–$(f4)$, we have that g is a Carathéodory function satisfying the following properties:

(g_1) $g(x,t) = 0$ for each $t \le 0$;

(g_2) $\lim\limits_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$, and there exists $q \in (4,6)$ such that

$$\lim_{t\to+\infty} \frac{g(x,t)}{t^{\frac{q-2}{2}}} = 0 \text{ uniformly in } x \in \mathbb{R}^3;$$

(g_3) $g(x,t) \le f(t)$ for all $t \ge 0$ and uniformly in $x \in \mathbb{R}^3$;

(g_4) $0 < \theta G(x,t) \le 2g(x,t)t$, for each $x \in \Lambda$, $t > 0$;

(g_5) $0 < G(x,t) \le g(x,t)t \le V_0 t/K$, for each $x \in \Lambda^c$, $t > 0$;

(g_6) for each $x \in \Lambda$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0,+\infty)$ and for each $x \in \Lambda^c$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0,a)$.

Then we consider the *modified* problem

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u + (|x|^{-1} * |u|^2)u = g(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{3.2}$$

Note that, if u is a solution of problem (3.2) with

$$|u(x)|^2 \le a \quad \text{for all } x \in \Lambda_\varepsilon^c, \quad \Lambda_\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon x \in \Lambda\},$$

then u is a solution of problem (2.2).

The functional associated to problem (3.2) is

$$J_\varepsilon(u) := \frac{1}{2}\int\limits_{\mathbb{R}^3} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx + \frac{1}{4}\int\limits_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 dx - \frac{1}{2}\int\limits_{\mathbb{R}^3} G(\varepsilon x, |u|^2)dx$$

defined in H_ε. It is standard to prove that $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and its critical points are the weak solutions of the modified problem (3.2).

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of J_ε, that is

$$\mathcal{N}_\varepsilon := \{u \in H_\varepsilon \backslash \{0\} : J_\varepsilon'(u)[u] = 0\},$$

and define the number c_ε by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Let H_ε^+ be open subset H_ε given by

$$H_\varepsilon^+ = \{u \in H_\varepsilon : |\text{supp}(u) \cap \Lambda_\varepsilon| > 0\},$$

and $S_\varepsilon^+ = S_\varepsilon \cap H_\varepsilon^+$, where S_ε is the unit sphere of H_ε. Note that S_ε^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_ε and contained in H_ε^+. Therefore, $H_\varepsilon = T_u S_\varepsilon^+ \bigoplus \mathbb{R}u$ for each $u \in T_u S_\varepsilon^+$, where $T_u S_\varepsilon^+ = \{v \in H_\varepsilon : \langle u, v\rangle_\varepsilon = 0\}$.

Now we show that the functional J_ε satisfies the Mountain Pass Geometry.

Lemma 3.1. *For any fixed $\varepsilon > 0$, the functional J_ε satisfies the following properties:*
(i) *there exist $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$;*
(ii) *there exists $e \in H_\varepsilon$ with $\|e\|_\varepsilon > r$ such that $J_\varepsilon(e) < 0$.*

Proof. (i) By (g_3), (f_1) and (f_2), for any $\zeta > 0$ small, there exists $C_\zeta > 0$ such that

$$G(\varepsilon x, |u|^2) \leq \zeta |u|^4 + C_\zeta |u|^q \quad \text{for all } x \in \mathbb{R}^3.$$

By the Sobolev embedding it follows

$$\begin{aligned} J_\varepsilon(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx - \frac{\zeta}{2}\int_{\mathbb{R}^3} |u|^4 dx - \frac{C_\zeta}{2}\int_{\mathbb{R}^3} |u|^q dx \\ &\geq \frac{1}{2}\|u_n\|_\varepsilon^2 - C_1\zeta\|u_n\|_\varepsilon^4 - C_2 C_\zeta \|u_n\|_\varepsilon^q. \end{aligned}$$

Hence we can choose some $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$ since $q > 4$.
(ii) For each $u \in H_\varepsilon^+$ and $t > 0$, by the definition of g and (f_3), one has

$$\begin{aligned} J_\varepsilon(tu) &\leq \frac{t^2}{2}\int_{\mathbb{R}^3} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx + \frac{t^4}{4}\int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx - \frac{1}{2}\int_{\Lambda_\varepsilon} G(\varepsilon x, t^2|u|^2)dx, \\ &\leq \frac{t^2}{2}\|u\|_\varepsilon^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx - C_1 t^\theta \int_{\Lambda_\varepsilon} |u|^\theta dx + C_2|\mathrm{supp}(u) \cap \Lambda_\varepsilon|. \end{aligned}$$

Since $\theta > 4$, we can get the conclusion. □

Since f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of S_ε^+.

Lemma 3.2. *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ are satisfied, then the following properties hold:*
$(A1)$*For any $u \in H_\varepsilon^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = J_\varepsilon(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞);*
$(A2)$*There is a $\tau > 0$ independent on u such that $t_u \geq \tau$ for all $u \in S_\varepsilon^+$. Moreover, for each compact $\mathcal{W} \subset S_\varepsilon^+$ there is such that $t_u \leq C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*
$(A3)$*The map $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\widehat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon = \widehat{m}_\varepsilon|_{S_\varepsilon^+}$ is a homeomorphism between S_ε^+ and $\mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$;*
$(A4)$*If there is a sequence $\{u_n\} \subset S_\varepsilon^+$ such that $dist(u_n, \partial S_\varepsilon^+) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

Proof. $(A1)$ As in the proof of Lemma 3.1, we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ verifying $g_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. From $(f4)$, the definition of g and $|\mathrm{supp}(u) \cap \Lambda_\varepsilon| > 0$, we may obtain the uniqueness of t_u. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a unique $t = t_u$ so that $g_u'(t) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$.

$(A2)$ For $\forall\, u \in S_\varepsilon^+$, we have

$$t_u + t_u^3 \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} g(\varepsilon x, t_u^2|u|^2) t_u |u|^2 dx.$$

From $(g2)$, the Sobolev embeddings and $q > 4$, we get

$$t_u \leq \zeta t_u^3 \int_{\mathbb{R}^3} |u|^4 dx + C_\zeta t_u^{q-1} \int_{\mathbb{R}^3} |u|^q dx \leq C_1 \zeta t_u^3 + C_2 C_\zeta t_u^{q-1},$$

which implies that $t_u \geq \tau$ for some $\tau > 0$. If $\mathcal{W} \subset S_\varepsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ with $t_n := t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists a $u \in \mathcal{W}$ such that $u_n \to u$ in H_ε. Moreover, using the proof of Lemma 3.1(ii), we have that $J_\varepsilon(t_n u_n) \to -\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_\varepsilon$, from $(g4)$, $(g5)$, $(g6)$ and $\theta > 4$, it yields that

$$\begin{aligned}
J_\varepsilon(v_n) =& J_\varepsilon(v_n) - \frac{1}{\theta} J'_\varepsilon(v_n)[v_n] \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|v_n\|_\varepsilon^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big)\int\limits_{\mathbb{R}^3} (|x|^{-1} \star |v_n|^2)|v_n|^2 dx \\
&+ \int\limits_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |v_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |v_n|^2)\Big) dx \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|v_n\|_\varepsilon^2 - \frac{1}{K}\int\limits_{\mathbb{R}^3} V(\varepsilon x)|v_n|^2 dx\Big) \\
\geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(1 - \frac{1}{K}\Big)\|v_n\|_\varepsilon^2.
\end{aligned}$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_\varepsilon = t_n$, we obtain

$$0 < \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(1 - \frac{1}{K}\Big) \leq \frac{J_\varepsilon(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This proves $(A2)$.

$(A3)$ First of all, we note that $\widehat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by $(A2)$, for each $u \in H_\varepsilon^+$, there is a unique $\widehat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$, then $u \in H_\varepsilon^+$. Otherwise, we have $|\operatorname{supp}(u) \cap \Lambda_\varepsilon| = 0$ and by $(g5)$ we have

$$\begin{aligned}
\|u\|_\varepsilon^2 \leq \|u\|_\varepsilon^2 + \int\limits_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx =& \int\limits_{\mathbb{R}^3} g(\varepsilon x, |u|^2)|u|^2 dx \\
=& \int\limits_{\Lambda_\varepsilon^c} g(\varepsilon x, |u|^2)|u|^2 dx \\
\leq& \frac{1}{K}\int\limits_{\mathbb{R}^3} V(\varepsilon x)|u|^2 dx \\
\leq& \frac{1}{K}\|u\|_\varepsilon^2
\end{aligned}$$

which is impossible since $K > 1$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$ is well defined and continuous. From

$$m_\varepsilon^{-1}\big(m_\varepsilon(u)\big) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{t_u\|u\|_\varepsilon} = u, \quad \forall u \in S_\varepsilon^+,$$

we conclude that m_ε is a bijection. Now we prove $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ such that $u_n \to u$ in H_ε. By $(A2)$, there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\varepsilon$, i.e.,

$$t_n^2\|u_n\|_\varepsilon^2 + t_n^2 \int\limits_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2 dx = \int\limits_{\mathbb{R}^3} g(\varepsilon x, t_n^2|u_n|^2)t_n^2|u_n|^2 dx, \quad \forall n \in N,$$

and passing to the limit as $n \to \infty$ in the last inequality, we obtain

$$t_0^2\|u\|_\varepsilon^2 + t_0^2 \int\limits_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx = \int\limits_{\mathbb{R}^3} g(\varepsilon x, t_0^2|u|^2)t_0^2|u|^2 dx,$$

which implies that $t_0 u \in \mathcal{N}_\varepsilon$ and $t_u = t_0$. This proves $\widehat{m}_\varepsilon(u_n) \to \widehat{m}_\varepsilon(u)$ in H_ε^+. Thus, $\widehat{m}_\varepsilon$ and m_ε are continuous functions and $(A3)$ is proved.

$(A4)$ Let $\{u_n\} \subset S_\varepsilon^+$ be a subsequence such that $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$, then for each $v \in \partial S_\varepsilon^+$ and $n \in N$, we have $|u_n| = |u_n - v|$ a.e. in Λ_ε. Therefore, by $(V1)$, $(V2)$ and the Sobolev embedding, there exists a constant $C_t > 0$ such that

$$\begin{aligned}\|u_n\|_{L^t(\Lambda_\varepsilon)} &\le \inf_{v\in\partial S_\varepsilon^+} \|u_n - v\|_{L^t(\Lambda_\varepsilon)}\\ &\le C_t\Big(\inf_{v\in\partial S_\varepsilon^+} \int_{\Lambda_\varepsilon} (|\nabla_{A_\varepsilon} u_n - v|^2 + V_\varepsilon(x)|u_n - v|^2)dx\Big)^{\frac{1}{2}}\\ &\le C_t \operatorname{dist}(u_n, \partial S_\varepsilon^+)\end{aligned}$$

for all $n \in N$, $t \in [2, 6]$. By $(g2)$, $(g3)$ and $(g5)$, for each $t > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} G(\varepsilon x, t^2|u_n|^2)dx &\le \int_{\Lambda_\varepsilon} F(t^2|u_n|^2)dx + \frac{t^2}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2 dx\\ &\le C_1 t^4 \int_{\Lambda_\varepsilon} |u_n|^4 dx + C_2 t^q \int_{\Lambda_\varepsilon} |u_n|^q dx + \frac{t^2}{K}\|u_n\|_\varepsilon^2\\ &\le C_3 t^4 \operatorname{dist}(u_n, \partial S_\varepsilon^+)^4 + C_4 t^q \operatorname{dist}(u_n, \partial S_\varepsilon^+)^q + \frac{t^2}{K}.\end{aligned}$$

Therefore,

$$\limsup_n \int_{\mathbb{R}^3} G(\varepsilon x, t^2|u_n|^2)dx \le \frac{t^2}{K}, \quad \forall t > 0.$$

On the other hand, from the definition of m_ε and the last inequality, for all $t > 0$, one has

$$\begin{aligned}\liminf_n J_\varepsilon(m_\varepsilon(u_n)) &\ge \liminf_n J_\varepsilon(tu_n)\\ &\ge \liminf_n \frac{t^2}{2}\|u_n\|_\varepsilon^2 - \frac{t^2}{K}\\ &= \frac{K-2}{2K}t^2,\end{aligned}$$

this implies that

$$\liminf_n \frac{1}{2}\|m_\varepsilon(u_n)\|_\varepsilon^2 \ge \frac{K-2}{2K}t^2, \quad \forall\, t > 0.$$

From the arbitrary of $t > 0$, it is easy to see that $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$ as $n \to \infty$. This completes the proof of Lemma 3.2. □

Now we define the function

$$\widehat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R},$$

by $\widehat{\Psi}_\varepsilon(u) = J_\varepsilon(\widehat{m}_\varepsilon(u))$ and denote by $\Psi_\varepsilon := (\widehat{\Psi}_\varepsilon)|_{S_\varepsilon^+}$.

We may obtain the following result from Lemma 3.2 directly, and its proof is similar to that of Corollary 10 in [30], so we omit it.

Lemma 3.3. *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ are satisfied, then*
$(B1)\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\varepsilon'(u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J_\varepsilon'(\widehat{m}_\varepsilon(u))[v], \quad \forall\, u \in H_\varepsilon^+ \text{ and } \forall\, v \in H_\varepsilon;$$

$(B2)\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ *and*

$$\Psi_\varepsilon'(u)v = \|m_\varepsilon(u)\|_\varepsilon J_\varepsilon'(\widehat{m}_\varepsilon(u))[v], \quad \forall v \in T_u S_\varepsilon^+;$$

(B3) *If $\{u_n\}$ is a $(PS)_c$ sequence of Ψ_ε, then $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence of J_ε. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_c$ sequence of J_ε, then $\{m_\varepsilon^{-1}(u_n)\}$ is a $(PS)_c$ sequence of Ψ_ε;*

(B4) *u is a critical point of Ψ_ε if and only if $m_\varepsilon(u)$ is a critical point of J_ε. Moreover, the corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon. = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon.$$

As in [30], we have the following variational characterization of the infimum of J_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in H_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu) = \inf_{u\in S_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu)$$

Lemma 3.4. *Let $c > 0$ and $\{u_n\}$ is a $(PS)_c$ sequence for J_ε, then $\{u_n\}$ is bounded in H_ε.*

Proof. Assume that $\{u_n\} \subset H_\varepsilon$ is a $(PS)_c$ sequence for J_ε, that is, $J_\varepsilon(u_n) \to c$ and $J'_\varepsilon(u_n) \to 0$. By using $(g4)$, $(g5)$ and $\theta > 4$, we have

$$\begin{aligned}
d + o_n(1) + o_n(1)\|u_n\|_\varepsilon \geq & J_\varepsilon(u_n) - \frac{1}{\theta} J'_\varepsilon(u_n)[u_n] \\
= & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) \int_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2 dx \\
& + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2} \int_{\Lambda_\varepsilon^c} G(\varepsilon x, |u_n|^2) dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2K} \int_{\mathbb{R}^3} V(\varepsilon x)|u_n|^2 dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta} - \frac{1}{2K}\Big)\|u_n\|_\varepsilon^2.
\end{aligned}$$

Since $K > \theta/(\theta - 2)$, from the above inequalities we obtain that $\{u_n\}$ is bounded in H_ε. ☐

The following result is important to prove the $(PS)_{c_\varepsilon}$ condition for the functional J_ε.

Lemma 3.5. *The functional J_ε satisfies the $(PS)_c$ condition at any level $c > 0$.*

Proof. Let $(u_n) \subset H_\varepsilon$ be a $(PS)_c$ for J_ε. By Lemma 3.4, (u_n) is bounded in H_ε. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_ε and $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ for all $1 \leq r < 6$ as $n \to +\infty$. Moreover, Lemma 2.1(ii) and the subcritical growth of g imply that $J'_\varepsilon(u) = 0$, and

$$\|u\|_\varepsilon^2 + \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} g(\varepsilon x, |u|^2)|u|^2 dx.$$

Let $R > 0$ be such that $\Lambda_\varepsilon \subset B_{R/2}(0)$. We show that for any given $\zeta > 0$, for R large enough,

$$\limsup_n \int_{B_R^c(0)} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2) dx \leq \zeta. \tag{3.3}$$

Let $\phi_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x \in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B_R^c(0), \quad 0 \leq \phi_R \leq 1, \quad \text{and} \quad |\nabla \phi_R| \leq C/R$$

where $C > 0$ is a constant independent of R. Since the sequence $(\phi_R u_n)$ is bounded in H_ε, we have

$$J'_\varepsilon(u_n)[\phi_R u_n] = o_n(1),$$

that is

$$\operatorname{Re}\int_{\mathbb{R}^3} \nabla_{A_\varepsilon} u_n \overline{\nabla_{A_\varepsilon}(\phi_R u_n)}dx + \int_{\mathbb{R}^3} V_\varepsilon(x)|u_n|^2\phi_R dx + \int_{\mathbb{R}^3}(|x|^{-1} \star |u_n|^2)|u_n|^2\phi_R dx$$
$$= \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2)|u_n|^2\phi_R dx + o_n(1).$$

Since $\overline{\nabla_{A_\varepsilon}(u_n\phi_R)} = i\overline{u_n}\nabla\phi_R + \phi_R\overline{\nabla_{A_\varepsilon} u_n}$, using $(g5)$, we have

$$\int_{\mathbb{R}^3}(|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx \le \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2)|u_n|^2\phi_R dx - \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\varepsilon} u_n \nabla\phi_R dx + o_n(1)$$
$$\le \frac{1}{K}\int_{\mathbb{R}^3} V_\varepsilon(x)|u_n|^2\phi_R dx - \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\varepsilon} u_n \nabla\phi_R dx + o_n(1).$$

By the definition of ϕ_R, the Hölder inequality and the boundedness of (u_n) in H_ε, we obtain

$$\left(1 - \frac{1}{K}\right)\int_{\mathbb{R}^3}(|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx \le \frac{C}{R}\|u_n\|_2\|\nabla_{A_\varepsilon} u_n\|_2 + o_n(1) \le \frac{C_1}{R} + o_n(1)$$

and so (3.3) holds.

Using $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$, for all $1 \le r < 6$ again, up to a subsequence, we have that

$$|u_n| \to |u| \text{ a.e. in } \mathbb{R}^3 \text{ as } n \to +\infty,$$

then

$$g(\varepsilon x, |u_n|^2)|u_n|^2 \to g(\varepsilon x, |u|^2)|u|^2 \text{ a.e. in } \mathbb{R}^3 \text{ as } n \to +\infty.$$

Moreover, from the subcritical growth of g and and the Lebesgue Dominated Convergence Theorem, we can infer

$$\lim_n \int_{B_R(0)} \left|g(\varepsilon x, |u_n|^2)|u_n|^2 - g(\varepsilon x, |u|^2)|u|^2\right| dx = 0.$$

Now, by $(g5)$ and (3.3) we have

$$\int_{B_R^c(0)} \left|g(\varepsilon x, |u_n|^2)|u_n|^2 - g(\varepsilon x, |u|^2)|u|^2\right| dx \le \frac{2}{K}\int_{B_R^c(0)}(|\nabla_{A_\varepsilon} u_n|^2 + V(\varepsilon x)|u_n|^2)dx < \frac{2\zeta}{K}$$

for every $\zeta > 0$.
Hence

$$\int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2)|u_n|^2 dx \to \int_{\mathbb{R}^3} g(\varepsilon x, |u|^2)|u|^2 dx \text{ as } n \to +\infty.$$

Finally, since $J'_\varepsilon(u) = 0$, we have

$$o_n(1) = J'_\varepsilon(u_n)[u_n] = \|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3}(|x|^{-1} \star |u_n|^2)|u_n|^2 dx - \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2)|u_n|^2 dx$$
$$= \|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3}(|x|^{-1} \star |u_n|^2)|u_n|^2 dx - \|u\|_\varepsilon^2 - \int_{\mathbb{R}^3}(|x|^{-1} \star |u|^2)|u|^2 dx + o_n(1).$$

Thus, from Lemma 2.1, the sequence (u_n) strong converges to u in H_ε and $\int_{\mathbb{R}^3}(|x|^{-1}\star|u_n|^2)|u_n|^2 dx \to \int_{\mathbb{R}^3}(|x|^{-1}\star |u|^2)|u|^2 dx$ as $n \to \infty$. □

Since f is only assumed to be continuous, the following result is required for the multiplicity result in the next section.

Corollary 3.1. *The functional Ψ_ε satisfies the $(PS)_c$ condition on S_ε^+ at any level $c > 0$.*

Proof. Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_c$ sequence for Ψ_ε. Then $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi_\varepsilon'(u_n)\|_\star \to 0$, where $\|\cdot\|_\star$ is the norm in the dual space $(T_{u_n}S_\varepsilon^+)^\star$. By Lemma 3.3($B3$), we know that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for J_ε in H_ε. From Lemma 3.5, we know that there exists a $u \in S_\varepsilon^+$ such that, up to a subsequence, $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in H_ε. By Lemma 3.2($A3$), we obtain

$$u_n \to u \text{ in } S_\varepsilon^+,$$

and the proof is complete. □

Proposition 3.1. *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ hold, then problem (3.2) has a ground state solution for any $\epsilon > 0$.*

Proof. Since

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u \in H_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu) = \inf_{u \in S_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu),$$

by the Ekeland variational principle [37], we obtain a minimizing $(PS)_{c_\varepsilon}$ sequence on S_ε^+ for the functional Ψ_ε. Moreover, by Corollary 3.1, we deduce the existence of a ground state $u \in H_\varepsilon$ for problem (3.2). □

4 Multiple solutions for the modified problem

4.1 The autonomous problem

For our scope, we need also to study the following *limit* problem

$$-\Delta u + V_0 u + (|x|^{-1} \star |u|^2)u = f(u^2)u, \quad u : \mathbb{R}^3 \to \mathbb{R}, \tag{4.1}$$

whose associated C^1-functional, defined in $H^1(\mathbb{R}^3, \mathbb{R})$, is

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_0 u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)dx.$$

Let

$$\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} : I_0'(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_0} I_0(u).$$

Let S_0 be the unit sphere of $H_0 := H^1(\mathbb{R}^3, \mathbb{R})$ and is complete and smooth manifold of codimension 1. Therefore, $H_0 = T_u S_0 \bigoplus \mathbb{R}u$ for each $u \in T_u S_0$, where $T_u S_0 = \{v \in H_0 : \langle u, v\rangle_0 = 0\}$.

Lemma 4.1. *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied, then the following properties hold:*

($a1$) *For any $u \in H_0 \backslash \{0\}$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_0(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞);*

($a2$) *There is a $\tau > 0$ independent on u such that $t_u > \tau$ for all $u \in S_0$. Moreover, for each compact $\mathcal{W} \subset S_0$ there is such that $t_u \leq C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*

($a3$) *The map $\widehat{m} : H_0 \backslash \{0\} \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous and $m_0 = \widehat{m}_0|_{S_0}$ is a homeomorphism between S_0 and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$.*

The proof of Lemma 4.1 is similar to that of Lemma 3.2, we omit it.

Lemma 4.2. *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied, then*
$(b1)\widehat{\Psi}_0 \in C^1(H_0\backslash\{0\}, \mathbb{R})$ *and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_0'(\widehat{m}(u))[v], \quad \forall u \in H_0\backslash\{0\} \text{ and } \forall v \in H_0;$$

$(b2)\Psi_0 \in C^1(S_0, \mathbb{R})$ *and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_0'(\widehat{m}(u))[v], \quad \forall v \in T_u S_0;$$

$(b3)$*If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_0*, then* $\{m(u_n)\}$ *is a* $(PS)_c$ *sequence of* I_0*. If* $\{u_n\} \subset \mathcal{N}_0$ *is a bounded* $(PS)_c$ *sequence of* I_0*, then* $\{m^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_0*;*
$(b4)u$ *is a critical point of* Ψ_0 *if and only if* $m(u)$ *is a critical point of* I_0*. Moreover, the corresponding critical values coincide and*

$$\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

The proof of Lemma 4.2 can be found in the proofs of Proposition 9 and Corollary 10 of Szulkin and Weth [30], so we omit it.

Similar to the previous argument, we have the following variational characterization of the infimum of I_0 over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u\in\mathcal{N}_0} I_0(u) = \inf_{u\in H_0\backslash\{0\}} \sup_{t>0} I_0(tu) = \inf_{u\in S_0} \sup_{t>0} I_0(tu)$$

The next result is useful in later arguments.

Lemma 4.3. *Let* $\{u_n\} \subset H_0$ *be a* $(PS)_c$ *sequence for* I_0 *such that* $u_n \rightharpoonup 0$*. Then, one of the following alternatives occurs:*
(i) $u_n \to 0$ *in* H_0 *as* $n \to +\infty$*;*
(ii) *there are a sequence* $\{y_n\} \subset \mathbb{R}^3$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \geq \beta.$$

Proof. Assume that (ii) does not hold. Then, for every $R > 0$, we have

$$\lim_n \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0.$$

Being $\{u_n\}$ bounded in H_0, by the Lion's lemma [37], it follows that

$$u_n \to 0 \text{ in } L^r(\mathbb{R}^3, \mathbb{R}), \; 2 < r < 6.$$

From the subcritical growth of f, we have

$$\int_{\mathbb{R}^3} F(u_n^2)dx = o_n(1) = \int_{\mathbb{R}^3} f(u_n^2)u_n^2 dx.$$

Moreover, from $I_0'(u_n)[u_n] \to 0$, it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_0 u_n^2)dx + \int_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} f(u_n^2)u_n^2 dx + o_n(1) = o_n(1).$$

Thus (i) holds. □

Remark 4.1. *From Lemma 4.3 we see that if u is the weak limit of $(PS)_{c_{V_0}}$ sequence $\{u_n\}$ of the functional I_0, then we have $u \neq 0$. Otherwise we have that $u_n \rightharpoonup 0$ and if $u_n \not\to 0$, from Lemma 4.3 it follows that there are a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \geq \beta > 0.$$

Then set $v_n(x) = u_n(x + z_n)$, it is easy to see that $\{v_n\}$ is also a $(PS)_{c_{V_0}}$ sequence for the functional I_0, it is bounded, and there exists $v \in H_0$ such that $v_n \rightharpoonup v$ in H_0 with $v \neq 0$.

Lemma 4.4. *Assume that V satisfies $(V1)$, $(V2)$ and f satisfies $(f1)$–$(f4)$, then problem (4.1) has a positive ground state solution.*

Proof. First of all, it is easy to show that $c_{V_0} > 0$. Moreover, if $u_0 \in \mathcal{N}_0$ satisfies $I_0(u_0) = c_{V_0}$, then $m^{-1}(u_0) \in S_0$ is a minimizer of Ψ_0, so that u_0 is a critical point of I_0 by Lemma 4.2. Now, we show that there exists a minimizer $u \in \mathcal{N}_0$ of $I_0|_{\mathcal{N}_0}$. Since $\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0 = c_{V_0}$ and S_0 is a C^1 manifold, by Ekeland's variational principle, there exists a sequence $\omega_n \subset S_0$ with $\Psi_0(\omega_n) \to c_{V_0}$ and $\Psi_0'(\omega_n) \to 0$ as $n \to \infty$. Put $u_n = m(\omega_n) \in \mathcal{N}_0$ for $n \in N$. Then $I_0(u_n) \to c_{V_0}$ and $I_0'(u_n) \to 0$ as $n \to \infty$ by Lemma 4.2$(b3)$. Similar to the proof of Lemma 3.4, it is easy to know that $\{u_n\}$ is bounded in H_0. Thus, we have $u_n \rightharpoonup u$ in H_0, $u_n \to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R})$, $1 \leq r < 6$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, thus $I_0'(u) = 0$. From Remark 4.1, we know that $u \neq 0$. Moreover, by Lemma 2.1,

$$\begin{aligned}
c_{V_0} &\leq I_0(u) = I_0(u) - \frac{1}{\theta} I_0'(u)[u] \\
&= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u\|_0^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} f(u^2)u^2 - \frac{1}{2}F(u^2)\Big) dx \\
&\leq \liminf_n \Big\{ \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_0^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) \int_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2 dx + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} f(u_n)u_n^2 - \frac{1}{2}F(u_n^2)\Big) dx \Big\} \\
&= \liminf_n \Big\{ I_0(u_n) - \frac{1}{\theta} I_0'(u_n)[u_n] \Big\} \\
&= c_{V_0},
\end{aligned}$$

thus, u is a ground state solution. From the assumption of f, $u \geq 0$, moreover, by [8, Proposition 6 and Proposition 7], we know that $u(x) > 0$ for $x \in \mathbb{R}^N$. The proof is complete. □

Lemma 4.5. *Let $(u_n) \subset \mathcal{N}_0$ be such that $I_0(u_n) \to c_{V_0}$. Then (u_n) has a convergent subsequence in H_0.*

Proof. Since $(u_n) \subset \mathcal{N}_0$, from Lemma 4.1$(a3)$, Lemma 4.2$(b4)$ and the definition of c_{V_0}, we have

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0, \quad \forall n \in N,$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{u \in S_0} \Psi_0(u).$$

Since S_0 is a complete C^1 manifold, by Ekeland's variational principle, there exists a sequence $\{\tilde{v}_n\} \subset S_0$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for Ψ_0 on S_0 and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Similar to the proof of Lemma 4.4, we may obtain the conclusion of this lemma. □

4.2 The technical results

In this subsection, we prove a multiplicity result for the modified problem (3.2) using the Ljusternik-Schnirelmann category theory. In order to get it, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^3, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \le t \le \delta/2$ and $\eta(t) = 0$ if $t \ge \delta$.
For any $y \in M$, let us introduce the function

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right) \exp\left(i\tau_y\left(\frac{\varepsilon x - y}{\varepsilon}\right)\right),$$

where

$$\tau_y(x) := \sum_i^3 A_i(y)x_i.$$

Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t\ge 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}).$$

Note that $t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$.
Let us define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as

$$\Phi_\varepsilon(y) := t_\varepsilon\Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.
Moreover, the energy of the above functions has the following behavior as $\varepsilon \to 0^+$.

Lemma 4.6. *The limit*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that the statement is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ satisfying

$$\left|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}\right| \ge \delta_0.$$

For simplicity, we write Φ_n, Ψ_n and t_n for $\Phi_{\varepsilon_n}(y_n)$, Ψ_{ε_n,y_n} and t_{ε_n}, respectively.
Similar to the proof of Lemma 3.4 in [36], by the Lebesgue Dominated Convergence Theorem, we have that

$$\|\Psi_n\|^2_{\varepsilon_n} \to \int_{\mathbb{R}^3} (|\nabla\omega|^2 + V_0\omega^2)dx \text{ as } n \to +\infty. \tag{4.2}$$

$$\int_{\mathbb{R}^3} (|x|^{-1} \star |\Psi_n|^2)|\Psi_n|^2 dx \to \int_{\mathbb{R}^3} (|x|^{-1} \star |\omega|^2)|\omega|^2 dx \quad \text{as } n \to +\infty. \tag{4.3}$$

Since $J'_{\varepsilon_n}(t_n\Psi_n)(t_n\Psi_n) = 0$, by the change of variables $z = (\varepsilon_n x - y_n)/\varepsilon_n$, observe that, if $z \in B_{\delta/\varepsilon_n}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$,we have

$$\begin{aligned}
\|\Psi_n\|^2_{\varepsilon_n} + t_n^2 \int_{\mathbb{R}^3} (|x|^{-1} \star |\Psi_n|^2)|\Psi_n|^2 dx &= \int_{\mathbb{R}^3} g(\varepsilon_n z + y_n, t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \\
&= \int_{\mathbb{R}^3} f(t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \\
&\ge \int_{B_{\delta/(2\varepsilon_n)}(0)} f(t_n^2\omega^2(z))\omega^2(z)dz \\
&\ge \int_{B_{\delta/2}(0)} f(t_n^2\omega^2(z))\omega^2(z)dz \\
&\ge f(t_n^2\gamma^2) \int_{B_{\delta/2}(0)} \omega^4(z)dz
\end{aligned}$$

for all n large enough and where $y = \min\{\omega(z) : |z| \le \delta/2\}$. Moreover, we have

$$t_n^{-2}\|\Psi_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^3}(|x|^{-1}\star|\Psi_n|^2)|\Psi_n|^2dx \ge \frac{f(t_n^2y^2)}{f(t_n^2y^2)}y^2\int_{B_{\delta/2}(0)}\omega^4(z)dz.$$

If $t_n \to +\infty$, by $(f4)$ we derive a contradiction.
Therefore, up to a subsequence, we may assume that $t_n \to t_0 \ge 0$.
If $t_n \to 0$, using the fact that f is increasing and the Lebesgue Dominated Convergence Theorem, we obtain that

$$\|\Psi_n\|_{\varepsilon_n}^2 + t_n^2\int_{\mathbb{R}^3}(|x|^{-1}\star|\Psi_n|^2)|\Psi_n|^2dx = \int_{\mathbb{R}^3}f(t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \to 0,\ \text{as } n\to+\infty,$$

which contradicts (4.2). Thus, from (4.2) and (4.3), we have $t_0 > 0$ and

$$\int_{\mathbb{R}^3(|\nabla\omega|^2+V_0\omega^2)dx+t_0^2\int_{\mathbb{R}^3}(|x|^{-1}\star|\omega|^2)|\omega|^2dx=\int_{\mathbb{R}^3}f(t_0\omega^2)\omega^2dx},$$

so that $t_0\omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_n \int_{\mathbb{R}^3}F(|t_n\Psi_n|^2)dx = \int_{\mathbb{R}^3}F(\omega^2)dx.$$

Hence

$$\lim_n J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(\omega) = c_{V_0}$$

which is a contradiction and the proof is complete. □

Now we define the barycenter map.
Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $Y : \mathbb{R}^3 \to \mathbb{R}^3$ defined by setting

$$Y(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \ge \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^3$ is defined by

$$\beta_\varepsilon(u) := \frac{1}{\|u\|_4^4}\int_{\mathbb{R}^3}Y(\varepsilon x)|u(x)|^4dx.$$

We have the following lemma.

Lemma 4.7. *The limit*

$$\lim_{\varepsilon\to0^+}\beta_\varepsilon(\Phi_\varepsilon(y)) = y$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that there exists $\kappa > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \ge \kappa. \tag{4.4}$$

Using the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3}(Y(\varepsilon_n z + y_n) - y_n)\eta^4(|\varepsilon_n z|)\omega^4(z)dz}{\int_{\mathbb{R}^3}\eta^4(|\varepsilon_n z|)\omega^4(z)dz}.$$

Taking into account $(y_n) \subset M \subset M_\delta \subset B_\rho$ and the Lebesgue Dominated Convergence Theorem, we can obtain that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.4). □

Now, we prove the following useful compactness result.

Proposition 4.1. *Let $\varepsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that the sequence $(|v_n|) \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^3, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.*

Proof. Since $J'_{\varepsilon_n}(u_n)[u_n] = 0$ and $J_{\varepsilon_n}(u_n) \to c_{V_0}$, arguing as in the proof of Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\varepsilon_n} \le C$ for all $n \in \mathbb{N}$.
Arguing as in the proof of Lemma 3.2 and recalling that $c_{V_0} > 0$, we have that there exist a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_n \int_{B_R(\tilde{y}_n)} |u_n|^2 dx \ge \beta. \tag{4.5}$$

Now, let us consider the sequence $\{|v_n|\} \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$. By the diamagnetic inequality (2.1), we get that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R})$, and using (4.5), we may assume that $|v_n| \rightharpoonup v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ for some $v \neq 0$.
Let now $t_n > 0$ be such that $\tilde{v}_n := t_n|v_n| \in \mathcal{N}_{V_0}$, and set $y_n := \varepsilon_n \tilde{y}_n$.
By the diamagnetic inequality (2.1), we have

$$c_{V_0} \le I_0(\tilde{v}_n) \le \max_{t\ge 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_0(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.
Since the sequences $\{|v_n|\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1(\mathbb{R}^3, \mathbb{R})$ and $|v_n| \not\to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, then (t_n) is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \ge 0$.
We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $(|v_n|)$ is bounded, we have $\tilde{v}_n \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, that is $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$.
Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, and, by Lemma 4.5, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^3, \mathbb{R})$.
Now we show the final part, namely that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for n large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \ge |y_n| - \varepsilon_n|x| > R.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, using $(V1)$ and the diamagnetic inequality (2.1), we get that

$$\begin{aligned}\int_{\mathbb{R}^3} (|\nabla|v_n||^2 + V_0|v_n|^2)dx &\le \int_{\mathbb{R}^3} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\ &\le \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} f(|v_n|^2)|v_n|^2 dx.\end{aligned} \tag{4.6}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\tilde{f}(t) \le V_0/K$, we can see that (4.6) yields

$$\min\left\{1, V_0\left(1 - \frac{1}{K}\right)\right\} \int_{\mathbb{R}^3} (|\nabla|v_n||^2 + |v_n|^2)dx = o_n(1),$$

that is $|v_n| \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which contradicts to $v \not\equiv 0$.
Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^3$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$

such that for every n large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$.
To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using the Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t\geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n)$, we obtain

$$\begin{aligned} c_{V_0} = I_0(\tilde{v}) &< \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla\tilde{v}|^2 + V(y_0)|\tilde{v}|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-1} \star |\tilde{v}|^2)|\tilde{v}|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}|^2)dx \\ &\leq \liminf_n\Big(\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla\tilde{v}_n|^2 + V(\varepsilon_n x + y_n)|\tilde{v}_n|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-1} \star |\tilde{v}_n|^2)|\tilde{v}_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|\tilde{v}_n|^2)dx\Big) \\ &= \liminf_n\Big(\frac{t_n^2}{2}\int_{\mathbb{R}^3}(|\nabla|u_n||^2 + V(\varepsilon_n z)|u_n|^2)dz + \frac{t_n^4}{4}\int_{\mathbb{R}^3}(|x|^{-1} \star |u_n|^2)|u_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(|t_n u_n|^2)dz\Big) \\ &\leq \liminf_n J_{\varepsilon_n}(t_n u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) = c_{V_0} \end{aligned}$$

which is impossible and the proof is complete. ☐

Let now

$$\tilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.
Fixed $y \in M$, since, by Lemma 4.6, $|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$, we get that $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$ small enough.

We have the following relation between $\tilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 4.8. *We have*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0^+$ as $n \to +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_n |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_0(t|u_n|) \leq J_{\varepsilon_n}(tu_n)$ for any $t \geq 0$. Therefore, recalling that $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can deduce that

$$c_{V_0} \leq \max_{t\geq 0} I_0(t|u_n|) \leq \max_{t\geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n) \tag{4.7}$$

which implies that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ as $n \to +\infty$.
Then, Proposition 4.1 implies that there exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n large enough.
Thus, making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3}(Y(\varepsilon_n z + y_n) - y_n)|u_n(z + \tilde{y}_n)|^4 dz}{\int_{\mathbb{R}^3} |u_n(z + \tilde{y}_n)|^4 dz}.$$

Since, up to a subsequence, $|u_n|(\cdot + \tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\varepsilon_n z + y_n \to y \in M$ for any $z \in \mathbb{R}^3$, we conclude. ☐

4.3 Multiplicity of solutions for problem (3.2)

Finally, we present a relation between the topology of M and the number of solutions of the modified problem (3.2).

Theorem 4.1. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, problem* (3.2) *has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions.*

Proof. For any $\epsilon > 0$, we define the function $\pi_\epsilon : M \to S_\varepsilon^+$ by

$$\pi_\epsilon(y) = m_\varepsilon^{-1}(\Phi_\epsilon(y)), \ \forall y \in M.$$

By Lemma 4.6 and Lemma 3.3($B4$), we obtain

$$\lim_{\epsilon\to 0} \Psi_\epsilon(\pi_\epsilon(y)) = \lim_{\epsilon\to 0} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \text{ uniformly in } y \in M.$$

Hence, there is a number $\hat{\epsilon} > 0$ such that the set $\tilde{S}_\varepsilon^+ := \{u \in S_\varepsilon^+ : \Psi_\varepsilon(u) \le c_{V_0} + h(\varepsilon)\}$ is nonempty, for all $\epsilon \in (0, \hat{\epsilon})$, since $\pi_\epsilon(M) \subset \tilde{S}_\varepsilon^+$. Here h is given in the definition of $\tilde{\mathcal{N}}_\varepsilon$.

Given $\delta > 0$, by Lemma 4.6, Lemma 3.2($A3$), Lemma 4.7, and Lemma 4.8, we can find $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \pi_\epsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and continuous. From Lemma 4.7, we can choose a function $\Theta(\epsilon, z)$ with $|\Theta(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$, for all $\epsilon \in (0, \hat{\epsilon})$ such that $\beta_\varepsilon(\Phi_\varepsilon(z)) = z + \Theta(\epsilon, z)$ for all $z \in M$. Define $H(t, z) = z + (1-t)\Theta(\epsilon, z)$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly, $H(0, z) = \beta_\varepsilon(\Phi_\varepsilon(z))$, $H(1, z) = z$ for all $z \in M$. That is, $H(t, z)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \pi_\epsilon$ and the embedding $\iota : M \to M_\delta$. Thus, this fact implies that

$$\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M)) \ge \mathrm{cat}_{M_\delta}(M). \tag{4.8}$$

By Corollary 3.1 and the abstract category theorem [30], Ψ_ε has at least $\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on S_ε^+. Therefore, from Lemma 3.3($B4$) and (4.8), we have that J_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $\tilde{\mathcal{N}}_\varepsilon$ which implies that problem (3.2) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions. □

5 Proof of Theorem 1.1

In this section we prove our main result. The idea is to show that the solutions u_ε obtained in Theorem 4.1 satisfy

$$|u_\varepsilon(x)|^2 \le a \text{ for } x \in \Lambda_\varepsilon^c$$

for ε small. The key ingredient is the following result.

Lemma 5.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of problem* (3.2) *for $\varepsilon = \varepsilon_n$. Then $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{|v_n|\}$ is bounded in $L^\infty(\mathbb{R}^N, \mathbb{R})$ and*

$$\lim_{|x|\to+\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

We use the Moser iteration method to prove the theorem. Although there is more one term for problem, by the calculation, it is easy to know this term does not affect the procedure. We may refer to [36] for the details, so we omit it for simplicity.

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$ and any $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ solution of problem (3.2), it holds

$$\|u_\varepsilon\|^2_{L^\infty(\Lambda^c_\varepsilon)} \leq a. \tag{5.1}$$

We argue by contradiction and assume that there is a sequence $\varepsilon_n \to 0$ such that for every n there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ which satisfies $J'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|^2_{L^\infty(\Lambda^c_{\varepsilon_n})} > a. \tag{5.2}$$

As in Lemma 5.1, we have that $J_{\varepsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $y_n := \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then, we can find $r > 0$, such that $B_r(y_n) \subset \Lambda$, and so $B_{r/\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ for all n large enough.
Using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \leq a$ in $B^c_R(0)$ and n large enough, where $v_n = u_n(\cdot + \tilde{y}_n)$. Hence $|u_n|^2 \leq a$ in $B^c_R(\tilde{y}_n)$ and n large enough. Moreover, if n is so large that $r/\varepsilon_n > R$, then $\Lambda^c_{\varepsilon_n} \subset B^c_{r/\varepsilon_n}(\tilde{y}_n) \subset B^c_R(\tilde{y}_n)$, which gives $|u_n|^2 \leq a$ for any $x \in \Lambda^c_{\varepsilon_n}$. This contradicts (5.2) and proves the claim.
Let now $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$, where $\tilde{\varepsilon}_\delta > 0$ is given by Theorem 4.1. Then we have $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions to problem (3.2). If $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ is one of these solutions, then, by (5.1) and the definition of g, we conclude that u_ε is also a solution to problem (2.2).
Finally, we study the behavior of the maximum points of $|\hat{u}_\varepsilon|$, where $\hat{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ is a solution to problem (1.1), as $\varepsilon \to 0^+$.
Take $\varepsilon_n \to 0^+$ and the sequence (u_n) where each u_n is a solution of (3.2) for $\varepsilon = \varepsilon_n$. From the definition of g, there exists $\gamma \in (0, a)$ such that

$$g(\varepsilon x, t^2)t^2 \leq \frac{V_0}{K} t^2, \quad \text{for all } x \in \mathbb{R}^N, \ |t| \leq \gamma.$$

Arguing as above we can take $R > 0$ such that, for n large enough,

$$\|u_n\|_{L^\infty(B^c_R(\tilde{y}_n))} < \gamma. \tag{5.3}$$

Up to a subsequence, we may also assume that for n large enough

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{5.4}$$

Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty < \gamma$. Thus, since $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$, using $(g5)$ and the diamagnetic inequality (2.1) that

$$\int_{\mathbb{R}^3} (|\nabla |u_n||^2 + V_0|u_n|^2) dx \leq \int_{\mathbb{R}^3} g(\varepsilon_n x, |u_n|^2)|u_n|^2 dx \leq \frac{V_0}{K} \int_{\mathbb{R}^3} |u_n|^2 dx$$

and, being $K > 1$, $\|u_n\| = 0$, which is a contradiction.
Taking into account (5.3) and (5.4), we can infer that the global maximum points p_n of $|u_{\varepsilon_n}|$ belongs to $B_R(\tilde{y}_n)$, that is $p_n = q_n + \tilde{y}_n$ for some $q_n \in B_R$. Recalling that the associated solution of problem (1.1) is $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point η_{ε_n} of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $q_n \in B_R$, $\varepsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of V allows to conclude that

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

□

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