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Sign-changing solutions for Schrödinger–Kirchhoff-type fourth-order equation with potential vanishing at infinity

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Abstract

The purpose of this paper is to study the existence of sign-changing solution to the following fourth-order equation:

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)\Delta u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^N, \tag{0.1}$$

where $5 \le N \le 7$, Δ^2 denotes the biharmonic operator, $K(x)$, $V(x)$ are positive continuous functions which vanish at infinity, and $f(u)$ is only a continuous function. We prove that the equation has a least energy sign-changing solution by the minimization argument on the sign-changing Nehari manifold. If, additionally, f is an odd function, we obtain that equation has infinitely many nontrivial solutions.

Keywords: Biharmonic operator; Sign-changing solution; Nonlocal term; Variational methods

1 Introduction and main results

This article is concerned with the following fourth-order Kirchhoff-type equation:

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)\Delta u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $5 \le N \le 7$, Δ^2 denotes the biharmonic operator, and a, b are positive constants.

When $a = 1, b = 0$, equation (1.1) becomes the following fourth-order equation (replace $\mathbb{R}^N$ with Ω):

$$\Delta^2 u - \Delta u + V(x)u = f(x, u), \quad x \in \Omega, \tag{1.2}$$

where Ω is an open subset of $\mathbb{R}^N$. There are many results focused on the existence, multiplicity, and concentration of solutions to problem (1.2), see for instance [12, 13, 45, 48, 53–55, 58, 59] and the references therein.

Problem (1.1) stems from the following Kirchhoff equation:

$$\Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a, b > 0$.

Problem (1.3) comes from the following equation:

$$u_{tt} - \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u), \quad x \in \Omega. \tag{1.4}$$

Because equation (1.4) is regarded as a good approximation for describing nonlinear vibrations of beams or plates, it is used to describe some phenomena that appear in different physical, engineering, and other sciences [5, 9].

Since it involves $(\int_{\mathbb{R}^N} |\nabla u|^2\,dx)\Delta u$ or $(\int_\Omega |\nabla u|^2\,dx)\Delta u$, problem (1.1) or (1.3) has become particularly interesting. By means of the fixed point theorems or variational approach, Ma [26–28] discussed positive solutions (or solutions) for problems similar to problem (1.3) when $\mathbb{N} = 1$. When $\mathbb{N} \geq 2$, there have been many papers about solutions to problem (1.1) or (1.3), see, for example, [4, 29, 36, 44, 46, 50–52]. However, except [21, 57], there are very few papers considering sign-changing solutions. By combining constraint variation methods and deformation lemma, Zhang et al. [57] studied sign-changing solution to problem (1.1) when $K(x) \equiv 1$. When $K(x) \equiv 1$ and $f(u) = |u|^{p-2}u, 4 < p < 2_*$ (2_* defined below), by the minimization argument on the sign-changing Nehari manifold, Khoutir and Chen [21] discussed sign-changing solution to problem (1.1).

It is noticed that in the past decades many mathematicians have paid much of their attention to nonlocal problems. The appearance of nonlocal terms in the equations not only marks their importance in many physical applications but also causes some difficulties and challenges from a mathematical point of view. Certainly, this fact makes the study of nonlocal problems particularly interesting. In addition to the equations of Kirchhoff type, there are also some nonlocal problems, such as Schrödinger–Poisson systems, equations with the fractional Laplacian operator, and so on. Especially, these days there is a good trend of existence of solutions for fractional-order differential equations which are definitely the generalized study [1, 2, 15–20, 22, 32, 33].

Throughout this paper, as in [3], we say that $(V,K) \in \mathcal{K}$ if continuous functions $V, K : \mathbb{R}^N \to \mathbb{R}$ satisfy the following conditions:

(VK_0) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$.

(VK_1) If $\{A_n\}_n \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leq R$ for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x) = 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

One of the following conditions occurs:

(VK_2) $K/V \in L^\infty(\mathbb{R}^N)$;

or

(VK_3) There is $p \in (2, 2_*)$ such that

$$\frac{K(x)}{|V(x)|^{\frac{2_*-p}{2_*-2}}} \to 0 \quad \text{as } |x| \to \infty,$$

where $2_* = \frac{2N}{N-4}$ is the critical Sobolev exponent.

As for the function f, we assume $f \in C(\mathbb{R}, \mathbb{R})$ and satisfies the following conditions:

(f_1)

$$f(t) = o\big(|t|^3\big) \quad \text{as } t \to 0 \text{ if } (VK_2) \text{ holds.}$$

(f_2)

$$\limsup_{t \to 0} \frac{f(t)}{|t|^{p-1}} < +\infty, \quad \text{if } (VK_3) \text{ holds.}$$

(f_3) f has a "quasicritical growth", that is,

$$\limsup_{t \to +\infty} \frac{f(t)}{|t|^{2_*-1}} = 0.$$

(f_4) $\lim_{t \to \infty} \frac{F(t)}{t^4} = +\infty$, where $F(t) = \int_0^t f(s)\,ds$.

(f_5) $\frac{f(t)}{|t|^3}$ is an increasing function of $t \in \mathbb{R} \backslash \{0\}$.

Let

$$A := \left\{u \in D^{1,2}\big(\mathbb{R}^N\big) : \int_{\mathbb{R}^N} V(x) u^2\,dx < \infty\right\}$$

be a Banach space endowed with the norm $\|u\|_A := (\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx)^{\frac{1}{2}}$.

It follows from $(V, K) \in \mathcal{K}$ that the space B given by

$$B := \left\{u \in D^{2,2}\big(\mathbb{R}^N\big) : \int_{\mathbb{R}^N} V(x) u^2\,dx < \infty\right\}$$

with

$$\|u\|_B := \left(\int_{\mathbb{R}^N} \big(|\Delta u|^2 + V(x)u^2\big)\,dx\right)^{\frac{1}{2}}$$

is compactly embedded into the weighted Lebesgue space $L_K^q(\mathbb{R}^N)$ for some $q \in (2, 2_*)$ (see Proposition 2.2), where $L_K^q(\mathbb{R}^N)$ given by

$$L_K^q\big(\mathbb{R}^N\big) := \left\{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^q\,dx < \infty\right\}$$

with

$$\|u\|_K := \left(\int_{\mathbb{R}^N} K(x)|u|^q\,dx\right)^{1/q}.$$

In this paper, we discuss our problem on the space

$$E := \left\{ u \in D^{1,2}(\mathbb{R}^N) \cap D^{2,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty \right\}.$$

So, it is easy to see that E is a Hilbert space. Furthermore,

$$(u,v) = \int_{\mathbb{R}^N} \bigl(\Delta u \Delta v + a\nabla u \cdot \nabla v + V(x)uv\bigr)\,dx, \qquad \|u\| = (u,u)^{\frac{1}{2}}.$$

For problem (1.1), the energy functional is given by

$$\begin{aligned} I_b(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \bigl(|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2\bigr)\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 \\ &\quad - \int_{\mathbb{R}^N} K(x)F(u)\,dx, \quad u \in E, \end{aligned}$$

where $F(u) = \int_0^u f(t)\,dt$.

It follows from the conditions of this paper that $I_b(u)$ belongs to C^1 and

$$\begin{aligned} \langle I_b'(u), v\rangle &= \int_{\mathbb{R}^N} \bigl(\Delta u \Delta v + a\nabla u \cdot \nabla v + V(x)uv\bigr)\,dx \\ &\quad + b\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\left(\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\,dx\right) - \int_{\mathbb{R}^N} K(x)f(u)v\,dx \end{aligned}$$

for any $u, v \in E$.

The solution of problem (1.1) is the critical point of the functional $I_b(u)$. Furthermore, we say that u is a sign-changing solution if $u \in E$ is a solution to problem (1.1) and $u^{\pm} \neq 0$, where $u^+ = \max\{u(x), 0\}$, $u^- = \min\{u(x), 0\}$.

As pointed out in the article, since the nonlocal term $(\int_{\mathbb{R}^N} |\nabla u|^2\,dx)\Delta u$ is involved, there is an essential difference between problem (1.1) and problem (1.2) when we discussed the existence of sign-changing solutions, see [6–8, 10, 25, 37, 61].

Therefore, to study sign-changing solutions for problem (1.1), as in [6, 7, 10], we first obtain a minimizer of I_b over the constraint

$$\mathcal{M} = \bigl\{ u \in E, u^{\pm} \neq 0 \text{ and } \langle I_b'(u), u^+\rangle = \langle I_b'(u), u^-\rangle = 0 \bigr\}.$$

The rest is to prove that the minimizer is a sign-changing solution of problem (1.1). It is noticed that there are some interesting results, for example, [11, 14, 23, 24, 30, 31, 34, 35, 38–43, 47, 56, 60], which considered sign-changing solutions for other nonlocal problems.

The main result can be stated as follows.

Theorem 1.1 *Suppose that $(V,K) \in \mathcal{K}$ and (f_1)–(f_5) are satisfied. Then problem (1.1) has a least energy sign-changing solution u in E. If, additionally, f is an odd function, then problem (1.1) has infinitely many nontrivial solutions.*

Remark 1.1 In this paper, the potential V vanishing at infinity means

$$\lim_{|x|\to+\infty} V(x) = 0,$$

which is also used to characterize one problem as zero mass.

The rest of this paper proceeds as follows. Sections 2 and 3 are devoted to our variational setting, and necessary lemmas are shown and proved, which shall be used in the proof of our main results in Sect. 4.

2 The variational framework and preliminary results

Proposition 2.1 ([13]) *Assume $(V,K)\in\mathcal{K}$. If (VK_2) holds, then B is continuously embedded in $L^q_K(\mathbb{R}^N)$ for every $q\in[2,2_*]$; if (VK_3) holds, then B is continuously embedded in $L^p_K(\mathbb{R}^N)$.*

Proposition 2.2 ([13]) *Assume $(V,K)\in\mathcal{K}$. If (VK_2) holds, then B is compactly embedded in $L^q_K(\mathbb{R}^N)$ for every $q\in(2,2_*)$; if (VK_3) holds, then B is compactly embedded in $L^p_K(\mathbb{R}^N)$.*

Remark 2.1 Since E is continuously embedded in B, it is easy to see that Proposition 2.1 and Proposition 2.2 also hold if B is replaced with E.

Lemma 2.1 ([13]) *Suppose that $(V,K)\in\mathcal{K}$ and (f_1)–(f_2) are satisfied. Let $\{v_n\}$ be a sequence such that $v_n\rightharpoonup v$ in E. Then*

$$\int_{\mathbb{R}^N} KF(v_n)\,dx\to\int_{\mathbb{R}^N} KF(v)\,dx,\qquad \int_{\mathbb{R}^N} Kf(v_n)v_n\,dx\to\int_{\mathbb{R}^N} Kf(v)v\,dx.$$

Lemma 2.2 *Assume that $(V,K)\in\mathcal{K}$ and (f_1)–(f_5) hold. Then, for any $u\in E\backslash\{0\}$,*

$$\lim_{|t|\to\infty}\int_{\mathbb{R}^N}\frac{Kf(tu)u}{t^3}=\infty.$$

Proof The proof is similar to that of Lemma 2.4 in [24], so we omit it here. □

Similarly, we have the following results.

Lemma 2.3 *Assume that $(V,K)\in\mathcal{K}$ and (f_1)–(f_5) hold. Then, for any $u\in E\backslash\{0\}$,*

$$\lim_{|t|\to\infty}\int_{\mathbb{R}^N}\frac{KF(tu)}{t^4}=\infty.$$

Lemma 2.4 *Assume that $(V,K)\in\mathcal{K}$ and (f_1)–(f_5) hold. Then, for any $u\in E\backslash\{0\}$,*

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\frac{Kf(tu)u}{t}=0.$$

Let

$$\mathcal{N}:=\bigl\{u\in E\backslash\{0\}:\bigl\langle I_b'(u),u\bigr\rangle=0\bigr\}.$$

The following results are very important, because they allow us to overcome the non-differentiability of $\mathcal{N}$ (see Lemma 2.5(iii)).

Lemma 2.5 *Assume that $(V,K)\in\mathcal{K}$ and (f_1)–(f_5) hold. If $u\in E$ with $u^{\pm}\neq 0$, then*

(i) *For each $u\in E\backslash\{0\}$, let $h_u:\mathbb{R}_+\to R$ be defined by $h_u(t)=I_b(tu)$. Then there is unique $t_u>0$ such that $h_u'(t)>0$ in $(0,t_u)$ and $h_u'(t)<0$ in (t_u,∞).*

(ii) *There is $\tau>0$ independent of u such that $t_u\geq\tau$ for all $u\in\mathcal{S}$, which is the unit sphere on E. Moreover, for each compact set $Q\subset\mathcal{S}$, there is $C_Q>0$ such that $t_u\leq C_Q$ for all $u\in Q$.*

(iii) *The map $\widehat{m}:E\backslash\{0\}\to\mathcal{N}$ given by $\widehat{m}(u)=t_uu$ is continuous and $m:=\widehat{m}_{|\mathcal{S}}$ is a homeomorphism between $\mathcal{S}$ and $\mathcal{N}$. Moreover, $m^{-1}(u)=u/\|u\|$.*

Proof If (VK_2) holds. From (f_1) and (f_3), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\bigl|f(s)\bigr|\leq\varepsilon|s|+C_\varepsilon|s|^{2_*-1},\quad s\in\mathbb{R}. \tag{2.1}$$

So,

$$\begin{aligned}I_b(tu)&=\frac{1}{2}\|tu\|^2+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2-\int_{\mathbb{R}^N}K(x)F(tu)\,dx\\&\geq\frac{t^2}{2}\|u\|^2+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2-\varepsilon\int_{\mathbb{R}^N}K(x)t^2u^2\,dx-C_\varepsilon\int_{\mathbb{R}^N}K(x)t^{2_*}u^{2_*}\,dx\\&\geq\left(\frac{1}{2}-\varepsilon|K/V|_\infty\right)t^2\|u\|^2+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2-C_\varepsilon|K|_\infty t^{2_*}\|u\|^{2_*}.\end{aligned} \tag{2.2}$$

Let $\varepsilon<\frac{1}{2}/|K/V|_\infty$, there is $t_0>0$ sufficiently small such that

$$h_u(t)=I_b(tu)>0,\quad\forall t<t_0. \tag{2.3}$$

If (VK_3) holds. According to arguments in [13], there is $C_p>0$ such that, for given $\varepsilon\in(0,C_p)$, there exists $L>0$ satisfying

$$\int_{\{x:|x|\geq L\}}K|u|^p\,dx\leq\varepsilon\int_{\{x:|x|\geq L\}}\bigl[V|u|^2+|u|^{2_*}\bigr]\,dx,\quad u\in E. \tag{2.4}$$

So, it follows from (f_2) and (f_3) that

$$I_b(tu)\geq\frac{t^2}{2}\|u\|^2+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2-C_1t^p\int_{\mathbb{R}^N}K(x)u^p\,dx-C_2t^{2_*}\int_{\mathbb{R}^N}K(x)|u|^{2_*}\,dx.$$

According to (2.4), Hölder's inequality, and (VK_0), one has that

$$\begin{aligned}I_b(tu)\geq{}&\frac{t^2}{2}\|u\|^2+\frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2-C_1\varepsilon t^p\int_{B_r^c(0)}C\bigl(V(x)u^2+|u|^{2_*}\bigr)\,dx\\&-C_1t^p|K|_{\frac{2_*}{2_*-p}(B_r(0))}\left(\int_{B_r(0)}u^{2_*}\,dx\right)^{\frac{p}{2_*}}-C_2|K|_\infty t^{2_*}\int_{\mathbb{R}^N}u^{2_*}\,dx\end{aligned}$$

$$\geq \frac{t^2}{2}\|u\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 - C_2|K|_\infty t^{2_*} C_{2_*}^{-\frac{2_*}{2}}\|u\|^{2_*}$$
$$- C_1\left(\varepsilon\|u\|^2 + \varepsilon C_{2_*}^{-\frac{2_*}{2}}\|u\|^{2_*} + C_{2_*}^{-\frac{p}{2}}|K|_{\frac{2_*}{2_*-p}(B_r(0))}\|u\|^{2_*}\right)t^p. \tag{2.5}$$

Since $p > 2$ and $2_* > 2$, we have that (2.3) also holds.

On the other hand, thanks to $F(s) \geq 0, \forall s \in \mathbb{R}$, we get

$$I_b(tu) \leq \frac{1}{2}\|tu\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 - \int_D K(x)F(tu)\,dx,$$

where $D \subset suppu$ is a measurable set with finite and positive measures.

Hence, by combining Fatou's lemma and (f_4), one has

$$\begin{aligned}\limsup_{t\to\infty}\frac{I_b(tu)}{\|tu\|^4} &\leq \limsup_{t\to\infty}\frac{1}{2\|tu\|^2} + \frac{b}{4\|tu\|^2} \\ &\quad - \liminf_{t\to\infty}\left\{\int_D K(x)\left[\frac{F(tu)}{\|tu\|^4}\right]\left(\frac{u}{\|u\|}\right)^4\right\} \\ &\to -\infty.\end{aligned} \tag{2.6}$$

So, there is $\widetilde{t}_0 > 0$ sufficiently large so that

$$h_u(t) = I_b(tu) > 0, \quad \forall t > \widetilde{t}_0. \tag{2.7}$$

Therefore, by the continuity of h_u and (f_5), there is $t_u > 0$ which is a maximum global point of h_u with $t_u u \in \mathcal{N}$.

We assert that t_u is the unique critical point of h_u. In fact, suppose, by contradiction, that there are $t_1 > t_2 > 0$ such that $h_u'(t_1) = h_u'(t_2) = 0$. Then

$$0 > \frac{1}{\|t_1u\|^2} - \frac{1}{\|t_2u\|^2} = \frac{1}{\|u\|^4}\int_{\mathbb{R}^N} K(x)\left[\frac{f(t_1u)}{\|t_1u\|^3} - \frac{f(t_2u)}{\|t_2u\|^3}\right]u^4\,dx \geq 0, \tag{2.8}$$

which is absurd.

Next, we prove (ii).

For any $u \in \mathcal{S}$, according to (i), there exists $t_u > 0$ such that

$$t_u^2\|u\|^2 + bt_u^4\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 = \int_{\mathbb{R}^N} K(x)f(t_uu)t_uu\,dx.$$

By (2.1), we have that

$$\begin{aligned}t_u^2\|u\|^2 \leq t_u^2\|u\|^2 + bt_u^4\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 &= \int_{\mathbb{R}^N} K(x)f(t_uu)t_uu\,dx \\ &\leq \int_{\mathbb{R}^N} K(x)\left[\varepsilon|t_uu|^2 + C_\varepsilon|t_uu|^{2_*}\right]dx \\ &\leq \varepsilon t_u^2\left\|KV^{-1}\right\|_\infty\|u\|^2 + t_u^{2_*}C_\varepsilon\|K\|_\infty\|u\|^{2_*}.\end{aligned} \tag{2.9}$$

So, there exists $\tau > 0$, independent of u, such that $t_u \geq \tau$.

On the other hand, let $Q \subset \mathcal{S}$ be compact. Suppose that there exist $\{u_n\} \subset Q, u \in Q$ such that $t_n := t_{u_n} \to \infty$, $u_n \to u$ in E. So, it follows from (2.6) that

$$I_b(t_n u_n) \to -\infty \quad \text{in } \mathbb{R}. \tag{2.10}$$

According to (f_5), we obtain that

$$f(t)t - 4F(t) \geq 0 \tag{2.11}$$

is increasing when $t > 0$ and decreasing when $t < 0$. Hence we have, for each $u \in \mathcal{N}$, that

$$\begin{aligned} I_b(u) &= I_b(u) - \frac{1}{4}\langle I_b'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^N} K(x)\big(f(u)u - 4F(u)\big)\,dx \geq 0. \end{aligned} \tag{2.12}$$

Thanks to $\{t_{u_n}u_n\} \subset \mathcal{N}$, replaced u with $t_{u_n}u_n$ in (2.12), from (2.10) we have a contradiction. Therefore, (ii) holds.

Finally, we prove (iii). We assert that m, $\widehat{m}$, m^{-1} are well defined. Indeed, for each $u \in E\backslash\{0\}$, by (i), there is unique $m(u) \in \mathcal{N}$. If $u \in \mathcal{N}$, then $u \neq 0$, it is easy to see that $m^{-1}(u) = u/\|u\| \in \mathcal{S}$. So, m^{-1} is well defined. Furthermore, we have that

$$\begin{aligned} m^{-1}\big(m(u)\big) &= m(t_u u) = \frac{t_u u}{t_u\|u\|} = u, \quad \forall u \in \mathcal{S}, \\ m\big(m^{-1}(u)\big) &= m\left(\frac{u}{\|u\|}\right) = t_{(\frac{u}{\|u\|})}\frac{u}{\|u\|} = u, \quad \forall u \in \mathcal{N}. \end{aligned}$$

Hence, m is bijective and m^{-1} is continuous.

In what follows, we prove $\widehat{m} : E\backslash\{0\} \to \mathcal{N}$ is continuous. Suppose $\{u_n\} \subset E\backslash\{0\}$ and $u \in E\backslash\{0\}$ such that $u_n \to u$ in E. According to (ii), there is $t_0 > 0$ such that $\|u_n\|t_{u_n} = t_{(\frac{u_n}{\|u_n\|})} \to t_0$. So, we have $t_{u_n} \to \frac{t_0}{\|u\|} =: \widetilde{t}_0$. Thanks to $t_{u_n}u_n \in \mathcal{N}$, one has that

$$t_{u_n}^2\|u_n\|^2 + bt_{u_n}^4\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx\right)^2 = \int_{\mathbb{R}^N} K(x)f(t_{u_n}u_n)t_{u_n}u_n\,dx.$$

From the above equality, let $n \to \infty$, one has that

$$\widetilde{t}_0^2\|u\|^2 + b\widetilde{t}_0^4\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2 = \int_{\mathbb{R}^N} K(x)f(\widetilde{t}_0 u)\widetilde{t}_0 u\,dx,$$

which indicates that $(t_0/\|u\|)u \in \mathcal{N}$ and $t_u = t_0/\|u\|$. Therefore, $\widehat{m}(u_n) \to \widehat{m}(u)$. So, the proof is completed. □

Define $\widehat{\Psi} : E \to \mathbb{R}$ and $\Psi : \mathcal{S} \to \mathbb{R}$ by

$$\widehat{\Psi}(u) = I_b\big(\widehat{m}(u)\big) \quad \text{and} \quad \Psi := \widehat{\Psi}|_{\mathcal{S}}.$$

By Lemma 2.5 and the result from [37], one has the following.

Proposition 2.3 *Assume that $(V,K)\in\mathcal{K}$ and (f_1)–(f_5) hold, then*
(i) $\widehat{\Psi}\in C^1(E\setminus\{0\},\mathbb{R})$ *and*

$$\langle\widehat{\Psi}'(u),v\rangle=\frac{\|\widehat{m}(u)\|}{\|u\|}\langle I_b'(\widehat{m}(u)),v\rangle,\quad \forall u\in E\setminus\{0\} \text{ and } \forall v\in E.$$

(ii) $\Psi\in C^1(\mathcal{S},\mathbb{R})$ *and*

$$\langle\Psi'(u),v\rangle=\|m(u)\|\langle I_b'(m(u)),v\rangle,\quad \forall v\in T_u\mathcal{S}.$$

(iii) *If u_n is a $(PS)_d$ sequence for Ψ, then $m(u_n)$ is a $(PS)_d$ sequence for I_b. If $u_n\subset\mathcal{N}$ is a bounded $(PS)_d$ sequence for I_b, then $m^{-1}(u_n)$ is a $(PS)_d$ sequence for Ψ.*
(iv) *u is a critical point of Ψ if, and only if, $m(u)$ is a nontrivial critical point of I_b. Moreover, corresponding critical values coincide and*

$$\inf_{\mathcal{S}}\Psi=\inf_{\mathcal{N}}I_b.$$

Proposition 2.4 *If (f_1)–(f_5) hold, then*

$$c_b:=\inf_{u\in\mathcal{N}}I_b(u)=\inf_{u\in E\setminus\{0\}}\max_{t>0}I_b(tu)=\inf_{u\in\mathcal{S}}\max_{t>0}I_b(tu)>0. \tag{2.13}$$

3 Technical lemmas

For $u\in E$ with $u^{\pm}\neq 0$, let $\varphi_u(s,t):=I_b(su^++tu^-), s>0, t>0$.

Lemma 3.1 *Assume that $(V,K)\in\mathcal{K}$ and (f_1)–(f_5) hold. If $u\in E$ with $u^{\pm}\neq 0$, then*

(i) *the pair (s,t) of a critical point of $\varphi_u(s,t)$ with $s,t>0$ if and only if $su^++tu^-\in\mathcal{M}$,*
(ii) *the map $\varphi_u(s,t)$ has a unique critical point (s_+,t_-), with $s_+=s_+(u)>0$ and $t_-=t_-(u)>0$, which is the unique maximum point of $\varphi_u(s,t)$.*
(iii) *The maps $\alpha_+(r)=\frac{\partial\varphi_u}{\partial s}(r,t_-)r$ and $\alpha_-(r)=\frac{\partial\varphi_u}{\partial t}(s_+,r)r$ are such that $\alpha_+(r)>0$ if $r\in(0,s_+)$, $\alpha_-(r)>0$ if $r\in(0,t_-)$, $\alpha_+(r)<0$ if $r\in(s_+,\infty)$, and $\alpha_-(r)<0$ if $r\in(t_-,\infty)$.*

Proof It is easy to see that

$$\begin{aligned}\nabla\varphi_u(s,t)&=\left(\langle I_b'(su^++tu^-),u^+\rangle,\langle I_b'(su^++tu^-),u^-\rangle\right)\\&=\left(\frac{1}{s}\langle I_b'(su^++tu^-),su^+\rangle,\frac{1}{t}\langle I_b'(su^++tu^-),tu^-\rangle\right)\\&:=\left(\frac{1}{s}g_u(s,t),\frac{1}{t}h_u(s,t)\right),\end{aligned}$$

where

$$\begin{aligned}g_u(s,t)&=s^2\|u^+\|^2+bs^4\left(\int_{\mathbb{R}^N}|\nabla u^+|^2\,dx\right)^2+bs^2t^2\left(\int_{\mathbb{R}^N}|\nabla u^+|^2\,dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^-|^2\,dx\right)\\&\quad-\int_{\mathbb{R}^N}K(x)f(su^+)su^+\,dx, \qquad (3.1)\\h_u(s,t)&=t^2\|u^-\|^2+bt^4\left(\int_{\mathbb{R}^N}|\nabla u^-|^2\,dx\right)^2+bs^2t^2\left(\int_{\mathbb{R}^N}|\nabla u^+|^2\,dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^-|^2\,dx\right)\end{aligned}$$

$$- \int_{\mathbb{R}^N} K(x) f\left(tu^-\right) tu^- \, dx. \tag{3.2}$$

Hence, item (i) is obvious.

In the following, we prove (ii). Firstly, we assert that $\mathcal{M} \neq \emptyset$. By (i), we only prove the existence of a critical point of $\varphi_u(s,t)$. Let $u \in E$ with $u^\pm \neq 0$ and $t_0 \geq 0$ fixed, it follows from (3.1) that

$$\begin{aligned} g_u(s,t_0) &= s^2\left(\left\|u^+\right\|^2 + bs^2\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)^2 + bt_0^2\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right) \right. \\ &\quad \left. - \int_{\mathbb{R}^N} \frac{K(x) f(su^+) u^+}{s} \, dx\right), \\ g_u(s,t_0) &= s^4\left(\frac{\|u^+\|^2}{s^2} + b\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)^2 + \frac{bt_0^2}{s^2}\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right) \right. \\ &\quad \left. - \int_{\mathbb{R}^N} \frac{K(x) f(su^+) u^+}{s^3} \, dx\right). \end{aligned}$$

Together with Lemma 2.4 and Lemma 2.2, one gets

$$g_u(s,t_0) > 0 \quad \text{for } s \text{ small enough}; g_u(s,t_0) < 0 \text{ for } s \text{ large enough}.$$

Since $g_u(s,t_0)$ is continuous, there exists $s_0 > 0$ such that $g_u(s_0,t_0) = 0$. We assert that s_0 is unique. In fact, supposing by contradiction, there exist $0 < s_1 < s_2$ such that $g_u(s_1,t_0) = g_u(s_2,t_0) = 0$, and then we have

$$\begin{aligned} &s_1^2\left\|u^+\right\|^2 + bs_1^4\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)^2 + bs_1^2 t_0^2\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right) \\ &\quad = \int_{\mathbb{R}^N} K(x) f\left(s_1 u^+\right) s_1 u^+ \, dx, \\ &s_2^2\left\|u^+\right\|^2 + bs_2^4\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)^2 + bs_2^2 t_0^2\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right) \\ &\quad = \int_{\mathbb{R}^N} K(x) f\left(s_1 u^+\right) s_1 u^+ \, dx. \end{aligned}$$

So,

$$\begin{aligned} &\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)\left[\left\|u^+\right\|^2 + bt_0^2\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right)\right] \\ &\quad = \int_{\mathbb{R}^N} K(x)\left[\frac{f(s_1 u^+)}{(s_1 u^+)^3} - \frac{f(s_2 u^+)}{(s_2 u^+)^3}\right]\left(u^+\right)^4 dx. \end{aligned}$$

Therefore, it follows from (f_5) and $0 < s_1 < s_2$ that we have a contradiction. That is, there exists unique $s_0 > 0$ such that $g_u(s_0,t_0) = 0$.

Let $\phi_1(t) := s(t)$, where $s(t)$ satisfies the properties just mentioned previously, with t in the place of t_0. Then the map $\phi_1 : \mathbb{R}_+ \to (0,\infty)$ is well defined.

By definition, one has that $\frac{\partial \varphi_u}{\partial s}(\phi_1(t),t) = 0t \geq 0$. Then

$$\phi_1(t)\left\|u^+\right\|^2 + b\phi_1(t)^3\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)^2 + b\phi_1(t)t^2\left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right)$$

$$= \int_{\mathbb{R}^N} K(x) f\big(\phi_1(t)u^+\big) u^+ \, dx, t \geq 0. \tag{3.3}$$

We assert that $\phi_1(t)$ has some good properties.

(1) $\phi_1(t)$ is continuous. To this end, let $t_n \to t_0$ as $n \to \infty$ and suppose, by contradiction, that there is a subsequence, still denoted by t_n, such that $\phi_1(t_n) \to \infty$.

Obviously, $\phi_1(t_n) \geq t_n$ for n large enough. According to (3.3), one has that

$$\begin{aligned} &\frac{\|u^+\|^2}{\phi_1(t_n)^2} + b\left(\int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx\right)^2 + \frac{bt_n^2}{\phi_1(t_n)^2}\left(\int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx\right)\left(\int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx\right) \\ &\quad = \int_{\mathbb{R}^N} K(x) \frac{f(\phi_1(t_n)u^+)}{(\phi_1(t_n)u^+)^3} (u^+)^4 \, dx. \end{aligned} \tag{3.4}$$

In view of Lemma 2.2, we have a contradiction. So $\phi_1(t_n)$ is bounded. Therefore, there exists $s_0 \geq 0$ such that, passing to a subsequence,

$$\phi_1(t_n) \to s_0. \tag{3.5}$$

Combining (3.3) with (3.5), we have that

$$\begin{aligned} &s_0 \|u^+\|^2 + bs_0^3\left(\int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx\right)^2 + bs_0 t_0^2\left(\int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx\right)\left(\int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx\right) \\ &\quad = \int_{\mathbb{R}^N} K(x) f\big(s_0 u^+\big) u^+ \, dx, \end{aligned}$$

that is,

$$\frac{\partial \varphi_u}{\partial s}(s_0, t_0) = 0.$$

Consequently, $s_0 = \phi_1(t_0)$. That is, ϕ_1 is continuous.

(2) $\phi_1(t) > 0$. Suppose, by contradiction, that there is a sequence $\{t_n\}$ such that $\phi_1(t_n) \to 0+$ as $n \to \infty$. In view of (3.3) and Lemma 2.4, we have

$$\|u^+\|^2 \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{K(x) f(\phi_1(t_n)u^+)u^+}{\phi_1(t_n)} \, dx = 0,$$

which is absurd, and hence there is $C > 0$ such that $\phi_1(t) \geq C$.

(3) $\phi_1(t) < t$ for t large. Indeed, if there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\phi_1(t_n) \geq t_n$ for all $n \in \mathbb{N}$, then arguing as in (3.4), we have a contradiction. Thus, $\phi_1(t) < t$ for t large.

Similarly, according to definition of $h_u(s,t)$, we can define a map $\phi_2 : \mathbb{R}_+ \to (0,\infty)$ by $\phi_2(s) = t(s)$ satisfying (1), (2), and (3).

By (3), there exists $C_1 > 0$ such that $\phi_1(t) \leq t$ and $\phi_2(s) \leq s$ respectively when $t, s > C_1$. Let $C_2 = \max\{\max_{t\in[0,C_1]} \phi_1(t), \max_{s\in[0,C_1]} \phi_2(s)\}$, $C = \max\{C_1, C_2\}$, define $T : [0, C] \times [0, C] \to \mathbb{R}_+^2$ by

$$T(s,t) = \big(\phi_1(t), \phi_2(s)\big).$$

It is easy to see that $T(s,t) \in [0,C] \times [0,C]$ for all $(s,t) \in [0,C] \times [0,C]$. Since T is continuous, using the Brouwer fixed point theorem, there exists $(s_+,t_-) \in [0,C] \times [0,C]$ such that

$$\bigl(\phi_1(t_-),\phi_2(s_+)\bigr) = (s_+,t_-). \tag{3.6}$$

It follows from $\phi_i > 0$ that $s_+,t_- > 0$. According to the definition, we have

$$\frac{\partial \varphi_u}{\partial s}(s_+,t_-) = \frac{\partial \varphi_u}{\partial t}(s_+,t_-) = 0.$$

We next shall prove the uniqueness of s_+,t_-. Suppose that $\omega \in \mathcal{M}$, one has

$$\begin{aligned}\nabla\varphi_\omega(1,1) &= \left(\frac{\partial \varphi_\omega}{\partial s}(1,1),\frac{\partial \varphi_\omega}{\partial t}(1,1)\right)\\ &= \bigl(\bigl\langle I_b'\bigl(\omega^+ + \omega^-\bigr),\omega^+\bigr\rangle,\bigl\langle I_b'\bigl(\omega^+ + \omega^-\bigr),\omega^-\bigr\rangle\bigr)\\ &= (0,0),\end{aligned}$$

which shows that $(1,1)$ is a critical point of φ_ω. Now, we need to prove that $(1,1)$ is the unique critical point of φ_ω with positive coordinates. Let (s_0,t_0) be a critical point of φ_ω such that $0 < s_0 \le t_0$. So, one has that

$$\begin{aligned}&s_0^2\bigl\|\omega^+\bigr\|^2 + bs_0^4\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^+\bigr|^2\,dx\right)^2 + bs_0^2t_0^2\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^+\bigr|^2\,dx\right)\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)\\ &\quad= \int_{\mathbb{R}^N}K(x)f\bigl(s_0\omega^+\bigr)s_0\omega^+\,dx\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&t_0^2\bigl\|\omega^-\bigr\|^2 + bt_0^4\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)^2 + bs_0^2t_0^2\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^+\bigr|^2\,dx\right)\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)\\ &\quad= \int_{\mathbb{R}^N}K(x)f\bigl(t_0\omega^-\bigr)t_0\omega^-\,dx.\end{aligned} \tag{3.8}$$

Thanks to $0 < s_0 \le t_0$ and (3.8), we have that

$$\begin{aligned}&\frac{\|\omega^-\|^2}{t_0^2} + b\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)^2 + b\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^+\bigr|^2\,dx\right)\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)\\ &\quad\ge \int_{\mathbb{R}^N}K(x)\left[\frac{f(t_0\omega^-)}{(t_0\omega^-)^3}\right]\bigl(\omega^-\bigr)^4\,dx.\end{aligned} \tag{3.9}$$

On the other hand, for $\omega \in \mathcal{M}$, we have

$$\begin{aligned}&\bigl\|\omega^-\bigr\|^2 + b\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)^2 + b\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^+\bigr|^2\,dx\right)\left(\int_{\mathbb{R}^N}\bigl|\nabla\omega^-\bigr|^2\,dx\right)\\ &\quad= \int_{\mathbb{R}^N}K(x)\left[\frac{f(\omega^-)}{(\omega^-)^3}\right]\bigl(\omega^-\bigr)^4\,dx.\end{aligned} \tag{3.10}$$

Combining (3.9) with (3.10), one has that

$$\left(\frac{1}{t_0^2}-1\right)\|\omega^-\|^2 \geq \int_{\mathbb{R}^N} K(x)\left[\frac{f(t_0\omega^-)}{(t_0\omega^-)^3}-\frac{f(\omega^-)}{(\omega^-)^3}\right](\omega^-)^4\,dx.$$

If $t_0 > 1$, the left-hand side of the above inequality is negative, which is absurd because the right-hand side is positive by condition (f_5). Therefore, we obtain that $0 < s_0 \leq t_0 \leq 1$.

Similarly, by (3.7) and $0 < s_0 \leq t_0$, we get

$$\left(\frac{1}{s_0^2}-1\right)\|\omega^+\|^2 \leq \int_{\mathbb{R}^N} K(x)\left[\frac{f(s_0\omega^+)}{(s_0\omega^+)^3}-\frac{f(\omega^+)}{(\omega^+)^3}\right](\omega^+)^4\,dx,$$

and from (f_5) this is absurd. Therefore, we have $s_0 \geq 1$. Consequently, $s_0 = t_0 = 1$, which indicates that $(1,1)$ is the unique critical point of φ_ω with positive coordinates.

Let $u \in E$, $u^\pm \neq 0$ and (s_1,t_1), (s_2,t_2) be the critical points of φ_u with positive coordinates. In view of (i), one has that

$$\omega_1 = s_1u^+ + t_1u^- \in \mathcal{M}, \qquad \omega_2 = s_2u^+ + t_2u^- \in \mathcal{M}.$$

So,

$$\omega_2 = \left(\frac{s_2}{s_1}\right)s_1u^+ + \left(\frac{t_2}{t_1}\right)t_1u^- = \left(\frac{s_2}{s_1}\right)\omega_1^+ + \left(\frac{t_2}{t_1}\right)\omega_1^- \in \mathcal{M}.$$

It follows from $\omega_1 \in E$ and $\omega_1^\pm \neq 0$ that $(\frac{s_2}{s_1},\frac{t_2}{t_1})$ is a critical point of the map φ_{ω_1} with positive coordinates. Thanks to $\omega_1 \in \mathcal{M}$, one has that

$$\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1.$$

Hence, $s_1 = s_2$, $t_1 = t_2$.

Now, we prove that the unique critical point is the unique maximum point of φ_u. In fact, using Lemma 2.3, we have that

$$\varphi_u(s,t) \to -\infty, \quad \big|(s,t)\big| \to \infty.$$

Hence, the maximum point of $\varphi_u(s,t)$ cannot be achieved on the boundary of $(\mathbb{R}_+ \times \mathbb{R}_+)$. Without loss of generality, we may assume that $(0,\bar{t})$ is a maximum point of $\varphi_u(s,t)$. But, according to Lemma 2.4, it is obvious that

$$\begin{aligned}\varphi_u(s,\bar{t}) = {} & \frac{s^2}{2}\|u^+\| + \frac{bs^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u^+|^2\,dx\right)^2 - \int_{\mathbb{R}^N}K(x)F(su^+)\,dx \\ & + \frac{\bar{t}^2}{2}\|u^-\| + \frac{b\bar{t}^4}{4}\left(\int_{\mathbb{R}^N}|\nabla u^-|^2\,dx\right)^2 - \int_{\mathbb{R}^N}K(x)F(\bar{t}u^-)\,dx \\ & + \frac{b}{2}s^2\bar{t}^2\left(\int_{\mathbb{R}^N}|\nabla u^+|^2\,dx\right)\left(\int_{\mathbb{R}^N}|\nabla u^-|^2\,dx\right)\end{aligned}$$

is an increasing function with respect to s if s is small enough. Hence, $(0,\bar{t})$ is not a maximum point of φ in $\mathbb{R}_+ \times \mathbb{R}_+$.

Finally, we prove (iii). From (i) of Lemma 2.5, we get $\frac{\partial\varphi_u}{\partial s}(r,t_-) > 0$ if $r \in (0,s_+)$ and $\frac{\partial\varphi_u}{\partial s}(s_+,t_-) = 0$ and $\frac{\partial\varphi}{\partial s}(r,t_-) < 0$ if $r \in (t_-,\infty)$. Therefore, α_+ and α_- have the same behavior. □

Lemma 3.2 *If $\{u_n\} \in \mathcal{M}$ and $u_n \rightharpoonup u$ in E, then $u \in E^{\pm}$.*

Proof For any $v \in \mathcal{M}$, we have that

$$\begin{aligned}\|v^{\pm}\|^2 &\le \|v^{\pm}\|^2 + b\left(\int_{\mathbb{R}^N}|\nabla v^{\pm}|^2\,dx\right)^2 + b\left(\int_{\mathbb{R}^N}|\nabla v^+|^2\,dx\right)\left(\int_{\mathbb{R}^N}|\nabla v^-|^2\,dx\right)\\ &= \int_{\mathbb{R}^N}K(x)f(v^+)v^+\,dx.\end{aligned}\tag{3.11}$$

Similar to (2.9), we obtain that there is $\tau > 0$ such that

$$\|v^{\pm}\| \ge \tau,\quad \forall v \in \mathcal{M}.\tag{3.12}$$

So, if $\{u_n\} \subset \mathcal{M}$, we have

$$\tau^2 \le \|u_n^{\pm}\|^2 \le \int_{\mathbb{R}^N}K(x)f(u_n^{\pm})u_n^{\pm}\,dx,\quad \forall n \in \mathbb{N}.\tag{3.13}$$

Combining $u_n \rightharpoonup u$ in E with Proposition 2.2, we have

$$\tau^2 \le \int_{\mathbb{R}^N}K(x)f(u^{\pm})u^{\pm}\,dx,\tag{3.14}$$

which shows that $u \in E^{\pm}$. □

Next, we consider the following minimization problem:

$$m := \inf\{I_b(u) : u \in \mathcal{M}\}.\tag{3.15}$$

We claim

$$m \ge 2c_b.\tag{3.16}$$

In fact, since $\mathcal{M} \subset \mathcal{N}$, we have $m \ge c_b$. On the other hand, for any $v \in \mathcal{M}$, according to (i) of Lemma 2.5, there exist positive constants s_+ and t_- such that $s_+v^+, t_-v^- \in \mathcal{N}$. Therefore, from (i) and (ii) of Lemma 3.1, we have

$$\begin{aligned}I_b(v) &\ge I_b(s_+v^+ + t_-v^-)\\ &\ge I_b(s_+v^+) + I_b(t_-v^-)\\ &\ge 2c_b, \forall v \in \mathcal{M}.\end{aligned}$$

Lemma 3.3 *Assume that $(V,K) \in \mathcal{K}$ and (f_1)–(f_5) hold, then m is achieved.*

Proof Let $\{u_n\}$ be a sequence in $\mathcal{M}$ such that

$$I_b(u_n) \to m. \tag{3.17}$$

We will show that u_n is bounded in E. In fact, suppose that there exists a subsequence that we still call u_n such that

$$\|u_n\| \to \infty. \tag{3.18}$$

Now, we define $v_n := u_n/\|u_n\|$ for all $n \in \mathbb{N}$. So, there exists $v \in E$ such that

$$v_n \rightharpoonup v \quad \text{in } E. \tag{3.19}$$

From Proposition 2.2, we conclude that, up to a subsequence,

$$v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{3.20}$$

Using (i) of Lemma 3.1, it follows from $\{u_n\} \subset \mathcal{M}$ that $s_+(v_n) = t_-(v_n) = \|u_n\|$. Therefore, using (i) of Lemma 3.1 again, we obtain

$$\begin{aligned} I\big(\|u_n\|v_n\big) \geq I(tv_n) = {}& \frac{t^2}{2}\|v_n\|^2 + \frac{(bt)^4}{4}\left(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\right)^2 \\ & - \int_{\mathbb{R}^N} K(x)F(tv_n)\,dx, \quad \forall t > 0, n \in \mathbb{N}. \end{aligned} \tag{3.21}$$

Let $t \geq 1$ in (3.21), we have that

$$I(u_n) \geq \frac{t^2}{2}\left(\|v_n\|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\right)^2\right) - \int_{\mathbb{R}^N} K(x)F(tv_n)\,dx, \quad \forall n \in \mathbb{N}. \tag{3.22}$$

Suppose that $v = 0$. Hence, from (3.19) and Lemma 2.1 we have

$$\int_{\mathbb{R}^N} KF(tv_n)\,dx \to 0, \quad \forall t > 0. \tag{3.23}$$

By (3.17) and (3.23), passing to the limit as $n \to \infty$ in (3.22), we have that

$$m \geq \frac{t^2}{2}\lim_{n\to\infty}\left(\|v_n\|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\right)^2\right), \quad \forall t \geq 1. \tag{3.24}$$

Thanks to $\|v_n\| = 1$, there exists a constant $\alpha_0 > 0$ such that

$$m \geq \frac{t^2}{2}\alpha_0, \quad \forall t \geq 1.$$

So, we have a contradiction. Hence, $v \neq 0$.

On the other hand, we get

$$\frac{I(u_n)}{\|u_n\|^4} = \frac{1}{2\|u_n\|^2} + \frac{b}{4\|u_n\|^4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx\right)^2 - \int_{\mathbb{R}^N} K(x)\frac{F(u_n)}{\|u_n\|^4}\,dx$$

$$\leq \frac{1}{2\|u_n\|^2} + \frac{b}{4} - \int_{\mathbb{R}^N} K(x) \frac{F(v_n\|u_n\|)}{(v_n\|u_n\|)^4} v_n^4 \, dx. \tag{3.25}$$

Thanks to $v \neq 0$, by using Lemma 2.3, we get

$$\int_{\mathbb{R}^N} K(x) \frac{F(v_n\|u_n\|)}{(v_n\|u_n\|)^4} v_n^4 \, dx \to +\infty. \tag{3.26}$$

Therefore, by (3.17), (3.18), and (3.26), passing to the limit as $n \to \infty$ in (3.25), we have a contradiction.

Hence, we deduce that $\{u_n\}$ is bounded in E. Therefore, there exists $u \in E$ such that $u_n \rightharpoonup u, u_n^{\pm} \rightharpoonup u^{\pm}$.

From Lemma 3.2, we have that $u \in E^{\pm}$. So, according to Lemma 3.1, there exist $s_+, t_- > 0$ such that

$$s_+ u^+ + t_- u^- \in \mathcal{M}. \tag{3.27}$$

We assert that

$$0 < s_+, \qquad t_- \leq 1. \tag{3.28}$$

In fact, according to Lemma 2.1, one has that

$$\int_{\mathbb{R}^N} Kf\left(u_n^{\pm}\right) u_n^{\pm} \, dx \to \int_{\mathbb{R}^N} Kf\left(u^{\pm}\right) u^{\pm} \, dx, \tag{3.29}$$

$$\int_{\mathbb{R}^N} KF\left((u_n)^{\pm}\right) dx \to \int_{\mathbb{R}^N} KF\left((u)^{\pm}\right) dx. \tag{3.30}$$

Since $u_n \rightharpoonup u$ in E, combining the continuous embedding $E \hookrightarrow D^{1,2}(\mathbb{R}^N)$ with the weak semicontinuity of the norm $\|u\|_{D^{1,2}} = (\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{\frac{1}{2}}$, we have

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \right) \geq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \tag{3.31}$$

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} \left|\nabla u_n^{\pm}\right|^2 dx \right) \geq \int_{\mathbb{R}^N} \left|\nabla u^{\pm}\right|^2 dx. \tag{3.32}$$

Thanks to $\{u_n\} \subset \mathcal{M}$, using (3.29), (3.32), and weak semicontinuity of the norm in E, we have

$$\left\langle I_b'(u), u^{\pm}\right\rangle \leq \liminf_{n\to\infty} \left\langle I_b'(u_n), u_n^{\pm}\right\rangle = 0. \tag{3.33}$$

Suppose that $0 < s_+ \leq t_-$, then from (3.27) we have that

$$\begin{aligned} &\frac{\|u^-\|^2}{t_-^2} + b\left(\frac{s_+}{t_-}\right)^2 \left(\int_{\mathbb{R}^N} \left|\nabla u^+\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right) + b\left(\int_{\mathbb{R}^N} \left|\nabla u^-\right|^2 dx\right)^2 \\ &\quad = \int_{\mathbb{R}^N} K(x) \left[\frac{f(t_- u^-)}{(t_- u^-)^3}\right] \left(u^-\right)^4 dx. \end{aligned} \tag{3.34}$$

By (3.33), we have that

$$\begin{aligned}
&\|u^-\|^2 + b\left(\int_{\mathbb{R}^N} |\nabla u^-|^2\,dx\right)^2 + b\left(\int_{\mathbb{R}^N} |\nabla u^+|^2\,dx\right)\left(\int_{\mathbb{R}^N} |\nabla u^-|^2\,dx\right) \\
&\quad \le \int_{\mathbb{R}^N} K(x)\left[\frac{f(u^-)}{(u^-)^3}\right](u^-)^4\,dx.
\end{aligned} \tag{3.35}$$

Combining (3.34) with (3.35), we have that

$$\begin{aligned}
&\left(\frac{1}{t_-^2} - 1\right)\|u^-\|^2 + b\left[\left(\frac{s_+}{t_-}\right)^2 - 1\right]\left(\int_{\mathbb{R}^N} |\nabla u^+|^2\,dx\right)\left(\int_{\mathbb{R}^N} |\nabla u^-|^2\,dx\right) \\
&\quad \ge \int_{\mathbb{R}^N} K(x)\left[\frac{t_- f(u^-)}{(t_- u^-)^3} - \frac{f(u^-)}{(u^-)^3}\right](u^-)^4\,dx.
\end{aligned}$$

From the above inequality and (f_5), we have that $0 < t_- \le 1$.

Now, we prove that $I_b(s_+u^+ + t_-u^-) = m$.

Denoting $\bar{u} := \bar{s}u^+ + \bar{t}u^-$. So, from (2.11), (3.27), (3.29), (3.30), and Fatou's lemma, we have that

$$\begin{aligned}
m &\le I_b(\bar{u}) - \frac{1}{4}\langle I_b'(\bar{u}), \bar{u}\rangle \\
&= \frac{1}{4}\|\bar{u}\|^2 + \frac{1}{4}\int_{\mathbb{R}^N} K(x)\big[f(\bar{u})\bar{u} - 4F(\bar{u})\big]\,dx \\
&= \frac{1}{4}\left(\|\bar{s}u^+\|^2 + \|\bar{t}u^-\|^2\right) + \frac{1}{4}\int_{\mathbb{R}^N} K(x)\big[f(\bar{s}u^+)(\bar{s}u^+) - 4F(\bar{s}u^+\big]\,dx \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N} K(x)\big[f(\bar{t}u^-)(\bar{t}u^-) - 4F(\bar{t}u^-)\big]\,dx \\
&\le \frac{1}{4}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^N} K(x)\big[f(u)u - 4F(u)\big]\,dx \\
&\le \liminf_{n\to\infty}\left[I_b(u_n) - \frac{1}{4}\langle I_b'(u_n), u_n\rangle\right] \\
&= m.
\end{aligned}$$

Consequently, $\bar{s} = \bar{t} = 1$. Thus, $\bar{u} = u$ and $I_b(u) = m$. □

4 The proof of the main results

In this section, we prove Theorem 1.1.

Proof First, we prove that the minimizer u for (3.15) is indeed a sign-changing solution of problem (1.1). If $I_b'(u) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|I_b'(v)\| \ge \theta \quad \text{for all } \|v - u\| \le 3\delta.$$

Choose $\sigma \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u\|}\})$. Let $\Omega := (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $\gamma(s,t) = su^+ + tu^-$, $(s,t) \in \Omega$. It follows from Lemma 3.1 that

$$\bar{m} := \max_{\partial\Omega} I_b \circ \gamma < m. \tag{4.1}$$

For $\varepsilon := \min\{(m-\bar{m})/2, \theta\delta/8\}$ and $S_\delta := B(u,\delta)$, according to Lemma 2.3 in [49], there is a deformation $\eta \in C([0,1]\times E, E)$ such that

(a) $\eta(t,v) = v$ if $v \notin I_b^{-1}([m-2\varepsilon, m+2\varepsilon]\cap S_{2\delta})$;

(b) $\eta(1, I_b^{m+\varepsilon}\cap S_\delta) \subset I_b^{m-\varepsilon}$;

(c) $I_b(\eta(1,v)) \le I_b(v)$ for all $v\in E$;

(d) $\|\eta(t,v)-v\| \le \delta$ for all $v\in E, t\in[0,1]$.

Firstly, we need to prove that

$$\max_{(s,t)\in\bar{\Omega}} I_b\bigl(\eta\bigl(1,\gamma(s,t)\bigr)\bigr) < m. \tag{4.2}$$

In fact, it follows from Lemma 3.1 that $I_b(\gamma(s,t)) \le m < m+\varepsilon$. That is,

$$\gamma(s,t) \in I_b^{m+\varepsilon}.$$

On the other hand, we have

$$\begin{aligned}\bigl\|\gamma(s,t)-u\bigr\|^2 &= \bigl\|(s-1)u^+ + (t-1)u^-\bigr\| \\ &\le 2\bigl((s-1)^2\bigl\|u^+\bigr\|^2 + (t-1)^2\bigl\|u^-\bigr\|^2\bigr) \\ &\le 2\sigma\|u\|^2 < \delta^2,\end{aligned}$$

which shows that $\gamma(s,t)\in S_\delta$ for all $(s,t)\in\bar{\Omega}$.

Therefore, according to (b), we have $I_b(\eta(1,\gamma(s,t))) < m-\varepsilon$. Hence, (4.2) holds.

In the following, we prove that $\eta(1,\gamma(\Omega))\cap\mathcal{M} \neq \varnothing$, which contradicts the definition of m.

Let $\xi(s,t) := \eta(1,\gamma(s,t))$ and

$$\begin{aligned}\Upsilon_0(s,t) &:= \bigl(\bigl\langle I_b'\bigl(\gamma(s,1)\bigr), su^+\bigr\rangle, \bigl\langle I_b'\bigl(\gamma(1,t)\bigr), tu^-\bigr\rangle\bigr) = \bigl(\bigl\langle I_b'\bigl(su^+ + u^-\bigr), su^+\bigr\rangle, \bigl\langle I_b'\bigl(u^+ + tu^-\bigr), tu^-\bigr\rangle\bigr), \\ \Upsilon_1(s,t) &:= \left(\frac{1}{s}\bigl\langle I_b'\bigl(\xi(s,1)\bigr), \bigl(\xi(s,1)\bigr)^+\bigr\rangle, \frac{1}{t}\bigl\langle I_b'\bigl(\xi(1,t)\bigr), \bigl(\xi(1,t)\bigr)^-\bigr\rangle\right).\end{aligned}$$

According to (iii) of Lemma 3.1, the C^1 function $\varphi_+(s) = \varphi_u(s,1)$ has a unique global maximum point $s^+ = 1$ (note that $s\varphi_+'(s) = I'(\gamma(s,1))su^+$). According to density, given $\varepsilon > 0$ small enough, there exists $\varphi_{+,\varepsilon}\in C^\infty([1-\sigma,1+\sigma])$ satisfying $\|\varphi_+ - \varphi_{+,\varepsilon}\|_{C^\infty([1-\sigma,1+\sigma])} < \varepsilon$ with $s^+ = 1$ being the unique maximum global point of $\varphi_{+,\varepsilon}$ in $[1-\sigma,1+\sigma]$. Hence, $\|\varphi_+ - \varphi_{+,\varepsilon}\|_{C^\infty([1-\sigma,1+\sigma])} < \varepsilon$, $\varphi_{+,\varepsilon}'(1) = 0$ and $\varphi_{+,\varepsilon}''(1) < 0$. Similarly, there exists $\varphi_{-,\varepsilon}\in C^\infty([1-\sigma,1+\sigma])$ satisfying $\|\varphi_- - \varphi_{-,\varepsilon}\|_{C^\infty([1-\sigma,1+\sigma])} < \varepsilon$, $\varphi_{-,\varepsilon}'(1) = 0$ and $\varphi_{-,\varepsilon}''(1) < 0$, where $\varphi_-(t) = \varphi_u(1,t)$.

Let $\Upsilon_\varepsilon \in C^\infty(\Omega)$ be defined by $\Upsilon_\varepsilon(s,t) = (s\varphi_{+,\varepsilon}'(s), t\varphi_{-,\varepsilon}'(t))$. Then we get $\|\Upsilon_\varepsilon - \Upsilon_0\|_{C(\Omega)} < \frac{3\sqrt{2}\varepsilon}{2}$, $(0,0)\notin\Upsilon_\varepsilon(\partial\Omega)$, and $(0,0)$ is a regular value of Υ_ε in Ω. On the other hand, $(1,1)$ is the unique solution of equation $\Upsilon_\varepsilon(t,s) = (0,0)$ in Ω. By using Brouwer's degree, for ε small enough, we have

$$\deg\bigl(\Upsilon_0,\Omega,(0,0)\bigr) = \deg\bigl(\Upsilon_\varepsilon,\Omega,(0,0)\bigr) = \operatorname{sgnJac}(\Upsilon_\varepsilon)(1,1).$$

From

$$\operatorname{Jac}(\Upsilon_\varepsilon)(1,1) = \bigl[\varphi_{+,\varepsilon}'(1) + \varphi_{+,\varepsilon}''(1)\bigr]\times\bigl[\varphi_{-,\varepsilon}'(1) + \varphi_{-,\varepsilon}''(1)\bigr] = \varphi_{+,\varepsilon}''(1)\times\varphi_{-,\varepsilon}''(1) > 0,$$

one has

$$\deg\bigl(\Upsilon_0,\Omega,(0,0)\bigr) = \operatorname{sgn}\bigl[\varphi''_{+,\varepsilon}(1)\times\varphi''_{-,\varepsilon}(1)\bigr] = 1, \tag{4.3}$$

where $\operatorname{Jac}(\Upsilon_\varepsilon)$ is the Jacobian determinant of Υ_ε and sgn denotes the sign function.

On the other hand, according to (4.2), one has

$$I\bigl(g(s,t)\bigr) \le \alpha < \frac{\alpha+m}{2} \le m-2\varepsilon, \quad \forall(t,s)\in\partial\Omega. \tag{4.4}$$

Combining (4.4) with item (a), we have that $\gamma=\xi$ on $\partial\Omega$. Therefore, $\Upsilon_1=\Upsilon_0$ on $\partial\Omega$ and

$$\deg\bigl(\Upsilon_1,\Omega,(0,0)\bigr) = \deg\bigl(\Upsilon_0,\Omega,(0,0)\bigr) = 1. \tag{4.5}$$

Therefore, we have $\Upsilon_1(s,t)=(0,0)$ for some $(t,s)\in\Omega$.

We claim that

$$\Upsilon_1(1,1) = \bigl(\bigl\langle I_b'\bigl(\xi(1,1)\bigr),\bigl(\xi(1,1)\bigr)^+\bigr\rangle,\bigl\langle I_b'\bigl(\xi(1,1)\bigr),\bigl(\xi(1,1)\bigr)^-\bigr\rangle\bigr) = (0,0). \tag{4.6}$$

If (4.6) holds, by $(1,1)\in\Omega$, we have that $\xi(1,1)=\eta(1,\gamma(1,1))\in\mathcal{M}$, this is $\eta(1,\gamma(\Omega))\cap\mathcal{M}\neq\varnothing$.

In what follows, we prove (4.6). If the zero (t,s) of Υ_1 obtained above is equal to $(1,1)$, there is nothing to do. If $(s,t)\neq(1,1)$, let $\delta_1=\max\{|t-1|,|s-1|\}$, $\Omega_1=(1-\delta_1/2,1+\delta_1/2)\times(1-\delta_1/2,1+\delta_1/2)$. So, $(s,t)\in\Omega\backslash\Omega_1$ and for getting $(s_1,t_1)\in\Omega_1$ such that $\Upsilon_1(s_1,t_1)=0$, we just repeat for Ω_1 as used in Ω. If $(s_1,t_1)=(1,1)$, there is nothing to do. Otherwise, we can continue with the argument and find in the nth step that (4.6) holds, or produce a sequence (s_n,t_n) which converges to $(1,1)$ such that $\Upsilon_1(s_n,t_n)$ and $(s_n,t_n)\in\Omega_{n-1}\backslash\Omega_n$ for all $n\in\mathbb{N}$ with $\Omega_0=\Omega$. Let $n\to\infty$ and, using the continuity of Υ_1, we have that (4.6) holds. That is,

$$\eta\bigl(1,\gamma(1,1)\bigr) = \xi(1,1)\in\mathcal{M}.$$

So, we obtain that $u:=u^++u^-$ is a critical point of I_b, that is, a sign-changing solution for problem (1.1).

Furthermore, if f is odd, the functional Ψ is even. Now we prove that Ψ satisfies the (PS) condition. From (2.4) and (3.16), we have that Ψ is bounded from below in $\mathcal{S}$. Suppose that $\{u_n\}\subset\mathcal{S}$ is a $(PS)_d$ sequence of Ψ, according to (iii) of Lemma 2.3, we know $\{v_n:=m(u_n)\}\subset\mathcal{N}$ is a $(PS)_d$ sequence of I_b on $\mathcal{N}$. Through the standard agrement at the beginning of this section, we know that v_n is bounded in E. So, there exists nonzero $v\in E$ such that

$$v_n\rightharpoonup v \quad \text{in } E, \qquad v_n\to v \quad \text{a.e. in } \mathbb{R}^N.$$

Therefore, we have that

$$\begin{aligned}\bigl\langle I'(v_n)-I'(v),v_n-v\bigr\rangle &= \|v_n-v\|^2 + b\biggl(\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\biggr)\biggl(\int_{\mathbb{R}^N}\nabla v_n\cdot\nabla(v_n-v)\,dx\biggr)\\ &\quad - \int_{\mathbb{R}^N}K(x)f(v_n)(v_n-v)\,dx + \int_{\mathbb{R}^N}K(x)f(v)(v_n-v)\,dx\end{aligned}$$

$$= o_n(1). \tag{4.7}$$

According to Lemma 2.1, we have that

$$\int_{\mathbb{R}^N} Kf(v_n)v_n\,dx \to \int_{\mathbb{R}^N} Kf(v)v\,dx.$$

We claim that

$$\int_{\mathbb{R}^N} Kf(v_n)v\,dx \to \int_{\mathbb{R}^N} Kf(v)v\,dx. \tag{4.8}$$

In fact, if (VK_2) holds, since

$$\left|\sqrt{K(x)}f(v_n)\chi_{\{|v_n\leq 1|\}}\right|^2 \leq CV(x)v_n^2,$$

where χ is a character function. Hence $\{\sqrt{K(x)}f(v_n)\chi_{\{|v_n\leq 1|\}}\}$ is bounded in $L^2(\mathbb{R}^N)$. Similarly, we have $\sqrt{K(x)}v \in L^2(\mathbb{R}^N)$. So, from $v_n \to v$a.e. in $\mathbb{R}^N$, we can get

$$\int_{\mathbb{R}^N} Kf(v_n)\chi_{\{|v_n\leq 1|\}}v\,dx \to \int_{\mathbb{R}^N} Kf(v)\chi_{\{|v\leq 1|\}}v\,dx. \tag{4.9}$$

On the other hand, since $|Kf(v_n)\chi_{\{|v_n\geq 1|\}}|^{\frac{2_*}{2_*-1}} \leq Cv_n^{2_*}$ and $v_n \to v$ a.e. on $\mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} Kf(v_n)\chi_{\{|v_n\geq 1|\}}v\,dx \to \int_{\mathbb{R}^N} Kf(v)\chi_{\{|v\geq 1|\}}v\,dx. \tag{4.10}$$

If (VK_3) holds, according to Proposition 2.2, $\int_{\mathbb{R}^N} K|v_n|^p\,dx < +\infty$ and

$$\left|K^{\frac{p-1}{p}}f(v_n)\chi_{\{|v_n\geq 1|\}}\right|^{\frac{p}{p-1}} \leq K|v_n|^p \leq \psi(x) \in L\big(\mathbb{R}^N\big).$$

Then, by similar discussion, we have that (4.9) and (4.10) hold.

Therefore, from the above discussion, we have that (4.8) holds. Similarly, we have

$$\int_{\mathbb{R}^N} Kf(v)v_n\,dx \to \int_{\mathbb{R}^N} Kf(v)v\,dx.$$

Since $v_n \rightharpoonup v$ in E and $E \subset D^{1,2}(\mathbb{R}^N)$, we get $v_n \rightharpoonup v$ in $D^{1,2}$. Then, by weak semicontinuity of the norm in $D^{1,2}(\mathbb{R}^N)$, we have that $b(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx)(\int_{\mathbb{R}^N} \nabla v_n \cdot \nabla(v_n - v)\,dx) \geq 0$. Therefore, according to (4.7), we get $v_n \to v$ in E. From Proposition 2.3, $\{u_n := m^{-1}(v_n)\} \subset \mathcal{S}$ and $u_n \to u = m^{-1}(v) \in \mathcal{S}$. That is, Ψ satisfies the Palais–Smale condition on $\mathcal{S}$. So, from Lemma 2.5, Proposition 2.3, and [37], the functional I_b has infinitely many critical points. □

5 Conclusions

In this paper, by the minimization argument on the sign-changing Nehari manifold and the quantitative deformation lemma, we discussed the existence of least energy sign-changing solution for a class of Schrödinger–Kirchhoff-type fourth-order equations with potential vanishing at infinity. Our results improve and generalize some interesting known results.

Since these days there is a good trend of existence of solutions for fractional-order differential equations which are definitely the generalized study, we will discuss some problems about fractional-order differential equations in the follow-up work.

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