A Chung–Feller property for the generalized Schröder paths☆

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ABSTRACT

In this paper, we consider the generalized Delannoy paths with steps $E = (1, 0)$, $D = (1, 1)$, $N = (0, 1)$, and $N' = (0, 2)$, where each step is labelled with weights 1, a, b, and d, respectively. By using Riordan array method to study enumeration of these paths in general case and with the restriction that no step goes above the main diagonal, we obtain three families of matrices. We consider the correlation between these matrices, and obtain a Chung–Feller type theorem for these paths. By way of illustration, we give several examples of Riordan arrays.

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1. Introduction

We consider lattice paths in the plane $\mathbb{Z} \times \mathbb{Z}$ consisting of steps $E = (1, 0)$, $D = (1, 1)$, $N = (0, 1)$, and $N' = (0, 2)$, where each step is labelled with weights 1, a, b, and d respectively. When $a = b = 1$ and $d = 0$, they are reduce to the Delannoy paths [6,8,19]. So we call these paths the generalized Delannoy paths. The weight of a path is the product of the weights of all its steps. Denote by $|\alpha|$ the weight of a path α. The weight of a set of paths is the sum of the total weights of all paths. Denote by $|S|$ the weight of a path set S. Let $G(n, k)$ be the set of all generalized Delannoy paths from $(0, 0)$ to $(k, n - k)$, and $G_{n,k} = |G(n, k)|$. Then, we obtain a infinite lower triangular array $(G_{n,k})_{n,k \in \mathbb{N}}$ in which the generalized Fibonacci numbers appear in the first column. Some special cases of these matrices have been studied in [4,21,22].

The generalized Delannoy paths whose ending point is on the diagonal $y = x$ are called generalized grand Schröder paths. A generalized Schröder path is a generalized grand Schröder path which never goes above the diagonal $y = x$. In Fig. 1, we give three examples of generalized Delannoy path, generalized grand Schröder path and generalized Schröder path. Let $U(n, k)$ be the set of all generalized grand Schröder paths from $(0, 0)$ to $(n, n - k)$, and $U_{n,k} = |U(n, k)|$, and let $V(n, k)$ be the set of all generalized Schröder paths from $(0, 0)$ to $(n, n - k)$, and $V_{n,k} = |V(n, k)|$. Then, we obtain two families of Riordan arrays $(U_{n,k})_{n,k \in \mathbb{N}}$ and $(V_{n,k})_{n,k \in \mathbb{N}}$. These families include the Catalan triangle, the Schröder matrix and some Riordan arrays recently studied (see [1,7,10,13,14,17,20,21]). When $a = d = 0$ and $b = 1$, we get the grand Dyck paths consisting of east steps $E = (1, 0)$ and north steps $N = (0, 1)$. The classical Chung–Feller theorem [3,5] says a property for grand Dyck paths: the number of grand Dyck paths from $(0, 0)$ to (n, n) containing exactly m north steps above the line $y = x$ is the same as the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ for any $m \in \{0, 1, 2, \ldots, n\}$, is independent of m. Woan [20] presents a new uniformly distributed parameter based on the rightmost lowest point of the paths from $(0, 0)$ to $(2n, 0)$ using steps $(1, 1)$ and $(1, -1)$.

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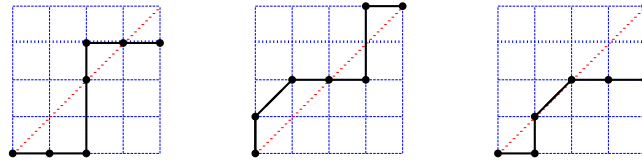*E-mail address:* slyang@lut.cn (S.-L. Yang).

Fig. 1. Examples of generalized Delannoy path, generalized grand Schröder path and generalized Schröder path.

In this paper, by considering the correlation between these two matrices $(U_{n,k})_{n,k\in\mathbb{N}}$ and $(V_{n,k})_{n,k\in\mathbb{N}}$, we obtain a Chung–Feller type theorem, that is, the set $U(n,0)\bigcup U(n,1)$ can be partitioned uniformly into $n+1$ blocks, and each block has size $V_{n,0} = |V(n,0)|$. This paper is organized as follows. In Section 2, we will recall the concept of Riordan matrix briefly and consider the enumeration of the generalized Delannoy paths with step set $\{E = (1,0), D = (1,1), N = (0,1), N' = (0,2)\}$. In Section 3, we enumerate the generalized grand Schröder paths and the generalized Schröder paths respectively. In Section 4, we prove a Chung–Feller property for the generalized Schröder paths. In the last section, we give more examples of Riordan arrays of combinatorial interest. Furthermore, we give a new interpretation of Catalan numbers.

2. Riordan array and a generalized Delannoy matrix

The concept of Riordan array was introduced in [15,18] as a generalization of the Pascal matrix. Recently, Riordan arrays have been used widely in the enumeration of lattice paths [4,9,11,12,18,21]. Here we briefly recall the notion of Riordan arrays. An infinite lower triangular matrix $R = (r_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $g(t)f(t)^k$, where $g(t)$ and $f(t)$ are formal power series with $g(0) = 1$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $g(t), f(t)$ is denoted by $R = (g(t), f(t))$. The set of all Riordan arrays forms a group under the ordinary row-by-column product with the multiplication identity $(1, t)$, called the Riordan group (see [9,15,18,24]). The multiplication rule of Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))), \tag{2.1}$$

and the inverse of $(g(t), f(t))$ is

$$(g(t), f(t))^{-1} = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right), \tag{2.2}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, i.e., $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

If $(b_n)_{n\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(g(t), f(t)) = (r_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} r_{n,k} b_k = [t^n] g(t) b(f(t)). \tag{2.3}$$

This is called the fundamental theorem of Riordan arrays and it can be rewritten as

$$(g(t), f(t))b(t) = g(t)b(f(t)). \tag{2.4}$$

For an infinite lower triangular matrix $R = (r_{n,k})_{n,k\in\mathbb{N}}$, the half of R is defined to be the infinite lower triangular matrix $(r_{2n-k,n})_{n,k\in\mathbb{N}}$. It is known that if $R = (p(t), tq(t))$ is a Riordan array, then its half is also a Riordan array [2,21–23].

Lemma 2.1. Let $R = (p(t), tq(t)) = (r_{n,k})_{n,k\geq 0}$ be a Riordan array and let $f(t)$ be the generating function defined by the functional equation $f(t) = tq(f(t))$. Then the half of Riordan array G is given by $H = \left(\frac{tf'(t)p(f(t))}{f(t)}, f(t)\right)$.

Let $G(n,k)$ be the set of all generalized Delannoy paths ending at the point $(k, n-k)$, $G_{n,k} = |G(n,k)|$ the weight of $G(n,k)$, and $G = G(a,b,d) = (G_{n,k})_{n,k\in\mathbb{N}}$. In Fig. 2, we give a schematic illustration of dependence of $G_{n+1,k+1}$ on the other elements in the array. From this, we get the recurrence:

$$G_{n+1,k+1} = G_{n,k} + bG_{n,k+1} + aG_{n-1,k} + dG_{n-1,k+1} \quad (n, k \geq 0) \tag{2.5}$$

and initial conditions are $G_{0,0} = 1$, $G_{1,0} = b$ and $G_{n+1,0} = bG_{n,0} + dG_{n-1,0}$.

The first few rows of the matrix $G(a,b,d) = (G_{n,k})_{n,k\in\mathbb{N}}$ are

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
b & 1 & 0 & 0 & 0 & \cdots \\
b^2 + d & a + 2b & 1 & 0 & 0 & \cdots \\
b^3 + 2bd & 2ab + 3b^2 + 2d & 2a + 3b & 1 & 0 & \cdots \\
b^4 + 3b^2d + d^2 & 3ab^2 + 4b^3 + 2ad + 6bd & a^2 + 6ab + 6b^2 + 3d & 3a + 4b & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Fig. 2. The matrix $(G_{n,k})$ and recurrence relation of its entries.

For $k \geq 0$, let $g_k(t) = \sum_{n=k}^{\infty} G_{n,k}t^n$. Then, from (2.5) we obtain that

$$g_{k+1}(t) = tg_k(t) + btg_{k+1}(t) + at^2 g_k(t) + dt^2 g_{k+1}(t).$$

Thus,

$$g_{k+1}(t) = \frac{t + at^2}{1 - bt - dt^2} g_k(t).$$

From this recurrence relation and the initial condition, we find that

$$g_0(t) = \sum_{n=k}^{\infty} G_{n,0}t^n = \frac{1}{1 - bt - dt^2},$$

and so

$$g_k(t) = \left(\frac{t + at^2}{1 - bt - dt^2} \right)^k \frac{1}{1 - bt - dt^2}.$$

Therefore, we proved the following theorem.

Theorem 2.2. *The matrix $G(a, b, d) = (G_{n,k})_{n,k \in \mathbb{N}}$ can be represented by Riordan array as*

$$G(a, b, d) = \left(\frac{1}{1 - bt - dt^2}, \frac{t + at^2}{1 - bt - dt^2} \right).$$

For example, the case $a = b = 1$ and $d = 0$ gives the Delannoy matrix

$$G(1, 1, 0) = \left(\frac{1}{1 - t}, \frac{t + t^2}{1 - t} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 & \cdots \\ 1 & 9 & 25 & 25 & 9 & 1 & 0 & \cdots \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In the case $a = b = d = 1$, the first few terms of the array $G(1, 1, 1)$ are

$$\left(\frac{1}{1 - t - t^2}, \frac{t + t^2}{1 - t - t^2} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 7 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 15 & 16 & 7 & 1 & 0 & 0 & \cdots \\ 8 & 30 & 43 & 29 & 9 & 1 & 0 & \cdots \\ 13 & 58 & 104 & 95 & 46 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.3. *The general term of the array $G(a, b, d)$ is*

$$G_{n,k} = \sum_{i=0}^{n-k} \sum_{j=0}^{k} \binom{i+k}{i}\binom{k}{j}\binom{i}{2i+j+k-n}a^j b^{2i+j+k-n}d^{n-k-i-j}. \tag{2.6}$$

Proof. From the definition and Theorem 2.2, we have

$$\begin{aligned}
G_{n,k} &= [t^n]\frac{1}{1-bt-dt^2}\left(\frac{t+at^2}{1-bt-dt^2}\right)^k \\
&= [t^n]\frac{t^k}{(1-bt-dt^2)^{k+1}}(1+at)^k \\
&= [t^{n-k}]\sum_{i=0}^{\infty}\binom{i+k}{i}(bt+dt^2)^i \sum_{j=0}^{k}\binom{k}{j}a^j t^j \\
&= [t^{n-k}]\sum_{i=0}^{\infty}\sum_{j=0}^{k}\binom{i+k}{i}\binom{k}{j}\sum_{p=0}^{i}\binom{i}{p}b^p d^{i-p}t^{2i-p}a^j t^j \\
&= [t^{n-k}]\sum_{i=0}^{\infty}\sum_{j=0}^{k}\sum_{p=0}^{i}\binom{i+k}{i}\binom{k}{j}\binom{i}{p}b^p d^{i-p}a^j t^{2i+j-p} \\
&= \sum_{i=0}^{n-k}\sum_{j=0}^{k}\binom{i+k}{i}\binom{k}{j}\binom{i}{2i+j+k-n}a^j b^{2i+j+k-n}d^{n-k-i-j}. \quad \square
\end{aligned}$$

3. Grand Schröder matrix and Schröder matrix

Let $U(n, k)$ be the set of all generalized grand Schröder paths ending at $(n, n-k)$ with no other restriction. Let $U_{n,k} = |U(n, k)|$, the sum of all $|\alpha|$ with α in $U(n, k)$. Then the array $U(a, b, d) = (U_{n,k})_{n,k\in\mathbb{N}}$ is the half of the matrix $G(a, b, d) = (G_{n,k})_{n,k\in\mathbb{N}}$, i.e, $U_{n,k} = G_{2n-k,n}$. We call $U(a, b, d) = (U_{n,k})_{n,k\in\mathbb{N}}$ the generalized grand Schröder matrix. The first few rows of $U(a, b, d)$ are

$$\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
a+2b & 1 & 0 & 0 & \cdots \\
a^2+6ab+6b^2+3d & 2a+3b & 1 & 0 & \cdots \\
a^3+12a^2b+30ab^2+20b^3+12ad+20bd & 3a^2+12ab+10b^2+4d & 3a+4b & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Theorem 3.1. *The matrix $U(a, b, d)$ is a Riordan array given by*

$$U(a, b, d) = \left(\frac{1}{1-at-2btV(t)-3dt^2V(t)^2}, tV(t)\right), \tag{3.1}$$

where $V(t)$ is determined by the equation $V(t) = 1 + atV(t) + btV(t)^2 + dt^2V(t)^3$, and

$$U_{n,k} = \sum_{i=0}^{n-k}\sum_{j=0}^{n}\binom{i+n}{i}\binom{n}{j}\binom{i}{2i+j-n+k}a^j b^{2i+j-n+k}d^{n-k-i-j}. \tag{3.2}$$

Proof. By definition, $U(a, b, d)$ is the half of $G(a, b, d) = \left(\frac{1}{1-bt-dt^2}, \frac{t+at^2}{1-bt-dt^2}\right)$. From Theorem 2.3, we get the formula for $U_{n,k}$. From Lemma 2.1, we obtain that $U(a, b, d)$ is a Riordan array and $U(a, b, d) = \left(\frac{tf'(t)}{f(t)(1-bf(t)-df(t)^2)}, f(t)\right)$, where $f(t)$ is the generating function defined by the functional equation $f(t) = \frac{t(1+af(t))}{1-bf(t)-df(t)^2}$. Now we denote $V(t) = \frac{f(t)}{t}$. Then $V(t)$ satisfies the equation $V(t) = 1 + atV(t) + btV(t)^2 + dt^2V(t)^3$ and $\frac{tf'(t)}{f(t)(1-bf(t)-df(t)^2)} = \frac{1}{1-at-2btV(t)-3dt^2V(t)^2}$. This completes the proof. $\square$

Recall that the generalized Delannoy paths ending at $(n, n-k)$ never going above the diagonal $y = x$ are called generalized Schröder paths. Let $V(n, k)$ be the set of all generalized Schröder paths ending at $(n, n-k)$, and $V_{n,k} = |V(n, k)|$. We call the matrix $V(a, b, d) = (V_{n,k})_{n,k\in\mathbb{N}}$ the generalized Schröder matrix.
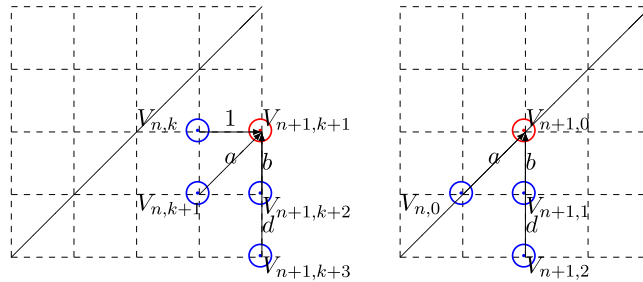
Fig. 3. The recurrence relation of entries of the generalized Schröder matrix $(V_{n,k})$.

The last step of any path from $V(n,k)$ is one of the step set $\{E = (1,0), D = (1,1), N = (0,1), N' = (0,2)\}$, as shown in Fig. 3.

Therefore, the numbers $V_{n,k}$ satisfy the following recurrence relations

$$V_{n+1,k+1} = V_{n,k} + aV_{n,k+1} + bV_{n+1,k+2} + dV_{n+1,k+3} \ (n > k \geq 0), \tag{3.3}$$

$$V_{n+1,0} = aV_{n,0} + bV_{n+1,1} + dV_{n+1,2} \ (n \geq 0), \tag{3.4}$$

$$V_{n,n} = 1 \ (n \geq 0), \quad V_{n,k} = 0 \ (k > n > 0). \tag{3.5}$$

The first rows of $V(a, b, d)$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ a+b & 1 & 0 & 0 & \cdots \\ a^2 + 3ab + 2b^2 + d & 2a + 2b & 1 & 0 & \cdots \\ a^3 + 6a^2b + 10ab^2 + 5b^3 + 4ad + 5bd & 3a^2 + 8ab + 5b^2 + 2d & 3a + 3b & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 3.2. *The generalized Schröder matrix $V(a, b, d) = (V_{n,k})_{n,k \in \mathbb{N}}$ is a Riordan array given by*

$$V(a, b, d) = (V(t), tV(t)),$$

where $V(t)$ is determined by the equation $V(t) = 1 + atV(t) + btV(t)^2 + dt^2V(t)^3$.

Proof. Let $(v_{n,k}) = (V(t), tV(t))$ in which $V(t)$ satisfies the equation

$$V(t) = 1 + atV(t) + btV(t)^2 + dt^2V(t)^3.$$

Then, by the definition of Riordan array, we have

$$\begin{aligned}
v_{n+1,0} &= [t^{n+1}]V(t) \\
&= [t^{n+1}](1 + atV(t) + btV(t)^2 + dt^2V(t)^3) \\
&= a[t^n]V(t) + (b[t^{n+1}]tV(t)^2 + d[t^{n+1}]t^2V(t)^3) \\
&= av_{n,0} + bv_{n+1,1} + dv_{n+1,2}, \\
v_{n+1,k+1} &= [t^{n+1}]V(t)(tV(t))^{k+1} \\
&= [t^{n+1}](1 + atV(t) + btV(t)^2 + dt^2V(t)^3)(tV(t))^{k+1} \\
&= [t^{n-k}]V(t)^{k+1} + a[t^{n-k-1}]V(t)^{k+2} + b[t^{n-k-1}]V(t)^{k+3} + d[t^{n-k-2}]V(t)^{k+4} \\
&= v_{n,k} + av_{n,k+1} + bv_{n+1,k+2} + dv_{n+1,k+3}.
\end{aligned}$$

This shows that $(v_{n,k})$ satisfy the same recurrence relation and the boundary conditions with the generalized Schröder matrix $(V_{n,k})$. Hence $(V_{n,k}) = (v_{n,k}) = (V(t), tV(t))$. This completes the proof. □

The referee kindly supplied a proof using A-matrix (see [12]) of a Riordan array. The proof goes like this. By theorem 3.1 in [12], the matrix $V(a, b, d) = (V_{n,k})_{n,k \in \mathbb{N}}$ is a Riordan array $(d(t), h(t))$. The generating functions of rows of the A-matrix are $P^{[0]}(t) = 1 + at$, and $P^{[i]}(t) = 0$ for all $i \geq 1$. The generating functions of associated sequences are $Q^{[1]}(t) = b + dt$, and $Q^{[i]}(t) = 0$ for all $i \geq 2$. Thus $h(t) = t(1 + ah(t)) + bh(t)^2 + dh(t)^3$, and consequently $h(t) = \frac{t(1+ah(t))}{1-bh(t)-dh(t)^2}$. From recurrence (3.4) and theorem 3.3 in [12], the generating function $d(t)$ satisfies the equation $d(t) = \frac{1}{1-at-bh(t)-dh(t)^2}$. It follows that $d(t) = \frac{h(t)}{t}$ and $d(t) = 1 + atd(t) + btd(t)^2 + dt^2d(t)^3$. Since $d(t)$ and $V(t)$ in Theorem 3.2 satisfy the same equation, then $d(t) = V(t)$. Hence, $V(a, b, d) = (V(t), tV(t))$.

Theorem 3.3. *The general term of the generalized Schröder matrix $V(a, b, d)$ is given by*

$$V_{n,k} = \frac{k+1}{n+1} \sum_{i=0}^{n+1} \sum_{j=0}^{n-k} \binom{n+1}{i} \binom{n+j}{j} \binom{j}{n-k-i-j} a^i b^{i+2j+k-n} d^{n-k-i-j}. \tag{3.6}$$

Proof. We have $V = \left(\frac{f(t)}{t}, f(t)\right)$, and $V^{-1} = \left(\frac{h(t)}{t}, h(t)\right)$, where $h(t) = \frac{t-bt^2-dt^3}{1+at}$. Using the definition of Riordan array and Lagrange inversion formula (see [6,21]), we obtain

$$\begin{aligned}
v_{n,k} &= [t^n]\frac{f(t)}{t}f(t)^k = [t^n]\frac{\bar{h}(t)}{t}\bar{h}(t)^k = [t^{n+1}]\bar{h}(t)^{k+1} = \frac{k+1}{n+1}[t^{n-k}]\left(\frac{t}{h(t)}\right)^{n+1} \\
&= [t^{n-k}]\frac{k+1}{n+1}(1+at)^{n+1}(1-bt-dt^2)^{-(n+1)} \\
&= [t^{n-k}]\frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}a^i t^i \sum_{j=0}^{\infty}\binom{n+j}{j}(bt+dt^2)^j \\
&= [t^{n-k}]\frac{k+1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}a^i t^i \sum_{j=0}^{\infty}\binom{n+j}{j}\sum_{q=0}^{j}\binom{j}{q}b^{j-q}d^q t^{j+q} \\
&= [t^{n-k}]\frac{k+1}{n+1}\sum_{j=0}^{\infty}\sum_{i=0}^{n+1}\sum_{q=0}^{j}\binom{n+1}{i}\binom{n+j}{j}\binom{j}{q}a^i b^{j-q}d^q t^{i+j+q} \\
&= \frac{k+1}{n+1}\sum_{i=0}^{n+1}\sum_{j=0}^{n-k}\binom{n+1}{i}\binom{n+j}{j}\binom{j}{n-k-i-j}a^i b^{i+2j+k-n}d^{n-k-i-j}.
\end{aligned}$$

This completes the proof. □

4. A Chung–Feller property

In this section, we will prove a Chung–Feller property for the generalized grand Schröder paths. First, we will prove the Chung–Feller type theorem by using the generating function method. Then, we give a combinatorial proof.

Theorem 4.1. *For $n \geq 0$, we have*

$$U_{n,0} + aU_{n,1} = (n+1)V_{n,0}.$$

Proof. By the proof of Theorem 3.3, $V_{n,0} = [t^n]\frac{1}{n+1}\left(\frac{1+at}{1-bt-dt^2}\right)^{n+1}$.

By the proof of Theorem 2.2, $G_{n,k} = [t^{n-k}]\frac{1}{1-bt-dt^2}\left(\frac{1+at}{1-bt-dt^2}\right)^k$. Hence,

$$\begin{aligned}
U_{n,0} &= G_{2n,n} = [t^n]\frac{1}{1-bt-dt^2}\left(\frac{1+at}{1-bt-dt^2}\right)^n, \\
U_{n,1} &= G_{2n-1,n} = [t^n]\frac{t}{1-bt-dt^2}\left(\frac{1+at}{1-bt-dt^2}\right)^n.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
U_{n,0} + aU_{n,1} &= [t^n]\frac{1}{1-bt-dt^2}\left(\frac{1+at}{1-bt-dt^2}\right)^n + [t^n]\frac{at}{1-bt-dt^2}\left(\frac{1+at}{1-bt-dt^2}\right)^n \\
&= [t^n]\frac{1+at}{1-bt-dt^2}\left(\frac{1+at}{1-bt-dt^2}\right)^n = [t^n]\left(\frac{1+at}{1-bt-dt^2}\right)^{n+1} \\
&= (n+1)V_{n,0}. □
\end{aligned}$$

In order to give a bijection proof of Theorem 4.1, we will introduce a special point for a generalized Delannoy path. By the definition, each generalized Delannoy path is coded by a word

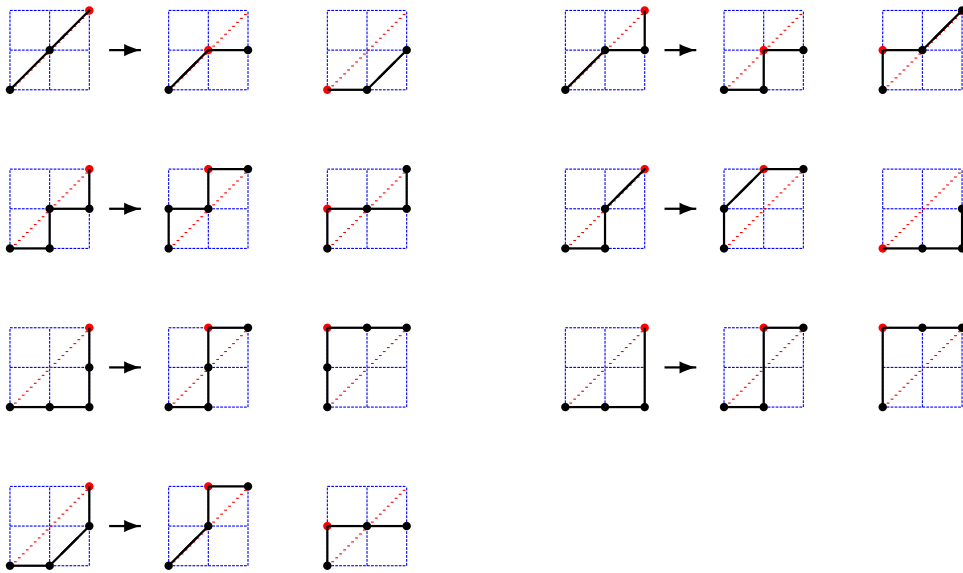$$\alpha = \alpha_1 \alpha_2 \cdots \alpha_m, \tag{4.1}$$

Fig. 4. The bijections from $H(0) = V(2, 0)$ to $H(1)$ and $H(2)$.

with each $\alpha_i \in \{E, D, N, N'\}$, the step set. Alternatively, every generalized Delannoy path can also be viewed as a sequence of lattice points. For instance, the path in (4.1) can also be written

$$\alpha = (a_0, b_0)(a_1, b_1) \ldots (a_m, b_m), \tag{4.2}$$

with $\alpha_i = (a_i - a_{i-1}, b_i - b_{i-1})$ for $i = 1, 2, \ldots, m$. For a given generalized Delannoy path $\alpha = (a_0, b_0)(a_1, b_1) \ldots (a_n, b_n)$, we define a *maximum point* (a_i, b_i) as a point in the path α such that $b_i - a_i \geq b_j - a_j$ for all $j = 0, 1, \ldots, n, j \neq i$. A *highest maximum point* (a_i, b_i) is a maximum point such that $b_i > b_j$ if (a_j, b_j) also is a maximum point and $j \neq i$. For example, in Figs. 4 and 5, the red point is the highest maximum point of the corresponding path.

Now we give a bijection proof of Theorem 4.1 in the case $a = b = d = 1$. We state it as the following theorem. In the coming proof, a path is regarded as a word of its steps.

Theorem 4.2. Let $a = b = d = 1$, $U^*(n) = U(n, 0) \bigcup U(n, 1)$, and $S' = \{E, D\}$. For $i = 0, 1, 2, \ldots, n$, let

$H(i) = \{\alpha \in U^*(n) : \text{there are } i \text{ steps from } S' \text{ after the highest maximum point of } \alpha\}$.

Then $H(0) = V(n, 0)$ and $\{H(i) : 0 \leq i \leq n\}$ uniformly partitions the set $U^*(n)$. Consequently, $U_{n,0} + U_{n,1} = (n + 1)V_{n,0}$.

Proof. That $H(0) = V(n, 0)$ follows directly from the definition. Let $\alpha \in H(0)$. For $1 \leq i \leq n$, write $\alpha = \beta X_i \gamma$, where β is an initial string, X_i is the ith appearance of alphabet in $S' = \{E, D\}$, and γ is a terminal string. We define a path $\varphi_i(\alpha)$ as (see Figs. 4 and 5)

$$\varphi_i(\alpha) = \gamma E \beta.$$

Then, $\varphi_i(\alpha) = \gamma E \beta$ has the starting point of E as the highest maximum point, since E moves the path right one unit and β is a initial segment of lattice path never going above the line $y = x$. Hence $\varphi_i(\alpha) \in H(i)$.

To show that the mapping φ_i is a bijection, we describe the inverse φ_i^{-1} as follows. Let $\alpha' \in H(i) \subseteq U(n, 0) \bigcup U(n, 1)$, where $i \neq 0$. Then we can decompose α' as $\alpha' = \gamma E \beta$, where the starting point of E is the highest maximum point of α', and there are i steps from S' after this point. We define a path $\varphi_i^{-1}(\alpha')$ as

$$\varphi_i^{-1}(\alpha') = \begin{cases} \beta E \gamma, & \text{if } \alpha' \in U(n, 0); \\ \beta D \gamma, & \text{if } \alpha' \in U(n, 1). \end{cases}$$

Hence, $\varphi_i^{-1}(\alpha') \in H(0)$. This completes the proof. $\square$

By the proof of Theorem 4.2, we can easily have the following result.

Corollary 4.3. *The total number of D steps in $V(n, 0)$ is equal to $U_{n,1}$, and the total number of E steps in $V(n, 0)$ is equal to $U_{n,0} - V_{n,0}$.*
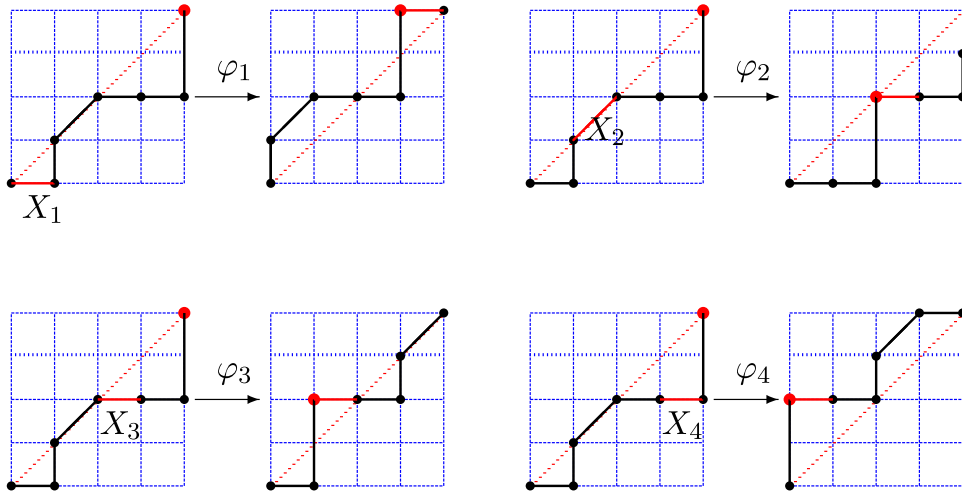
Fig. 5. A path $\alpha \in H(0) = V(4, 0)$ and $\varphi_i(\alpha) \in H(i)$, $i = 1, 2, 3, 4$.

Bijective proof of Theorem 4.1. In the proof of Theorem 4.2, we have established a bijection φ_i between $V(n, 0)$ and $H(i)$ by $\varphi_i(\alpha) = \gamma E \beta$ when $\alpha = \beta X_i \gamma$ with $X_i \in S' = \{E, D\}$, $i = 1, 2, \ldots, n$. When the steps are weighted, the weight $|\varphi_i(\alpha)|$ is related to the weight $|\alpha|$ as follows

$$|\varphi_i(\alpha)| = \begin{cases} |\alpha|, & \text{if } X_i = E; \\ \frac{1}{a}|\alpha|, & \text{if } X_i = D, \end{cases} \tag{4.3}$$

or equivalently,

$$|\varphi_i^{-1}(\alpha')| = \begin{cases} |\alpha'|, & \text{if } \alpha' \in U(n, 0); \\ a|\alpha'|, & \text{if } \alpha' \in U(n, 1). \end{cases} \tag{4.4}$$

Let $V(n) = V(n, 0) \times \{0, 1, 2, \ldots, n\}$. An element (α, i) of $V(n)$ is called a labelled path and its weight is defined as the weight of α. Then, the weight of $V(n)$ is

$$|V(n)| = (n + 1)|V(n, 0)| = (n + 1)V_{n,0}.$$

On the other hand, we define a map ψ from $U^*(n) = U(n, 0) \bigcup U(n, 1)$ to $V(n) = V(n, 0) \times \{0, 1, 2, \ldots, n\}$ such that for every $\alpha' \in H(i)$, where $U^*(n) = H(0) \bigcup H(1) \cdots H(n)$,

$$\psi(\alpha') = (\varphi_i^{-1}(\alpha'), i), \quad i = 0, 1, \ldots, n,$$

where φ_0 is regarded as the identity map from $V(n, 0)$ to $H(0) = V(n, 0)$.

Clearly, ψ is a bijection. Hence, $|V(n)| = |\psi(U^*(n))| = |\psi(U(n, 0))| + |\psi(U(n, 1))| = U_{n,0} + aU_{n,1}$. Therefore, we have $U_{n,0} + aU_{n,1} = (n + 1)V_{n,0}$, as desired. $\square$

If $a = d = 0, b = 1$, then

$$U(0, 1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 3 & 1 & 0 & 0 & \cdots \\ 20 & 10 & 4 & 1 & 0 & \cdots \\ 70 & 35 & 15 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad V(0, 1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The first column of the matrix $U(0, 1, 0)$ corresponds to the central binomial coefficients $\binom{2n}{n}$ (sequence A000984 in [16]), and that of $V(0, 1, 0)$ corresponds to the Catalan numbers C_n (sequence A000108 in [16]). Thus, by Theorem 4.1, we have

$$C_n = V_{n,0} = \frac{1}{n + 1}U_{n,0} = \frac{1}{n + 1}\binom{2n}{n}.$$

This result is the famous Chung–Feller Theorem.

If $d = 0, a = b = 1$, then

$$
U(1, 1, 0) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
3 & 1 & 0 & 0 & 0 & \cdots \\
13 & 5 & 1 & 0 & 0 & \cdots \\
63 & 25 & 7 & 1 & 0 & \cdots \\
321 & 129 & 41 & 9 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad
V(1, 1, 0) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 0 & \cdots \\
6 & 4 & 1 & 0 & 0 & \cdots \\
22 & 16 & 6 & 1 & 0 & \cdots \\
90 & 68 & 30 & 8 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The first column of the matrix $U(1, 1, 0)$ corresponds to the central Delannoy numbers (sequence A001850 in [16]), and that of $V(1, 1, 0)$ corresponds to the Schröder numbers (sequence A006318 in [16]). Thus, by Theorem 3.1, we have

$$(n + 1)V_{n,0} = U_{n,0} + U_{n,1}.$$

5. More examples

Example 5.1. For $a = b = d = 1$, the first few rows of the matrices $U(1, 1, 1)$ and $V(1, 1, 1)$ are

$$
U(1, 1, 1) = \left(\frac{1 - 2t - 4t^2 - 2t^3}{1 + t}, \frac{t - t^2 - t^3}{1 + t} \right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
3 & 1 & 0 & 0 & 0 & \cdots \\
16 & 5 & 1 & 0 & 0 & \cdots \\
95 & 29 & 7 & 1 & 0 & \cdots \\
591 & 179 & 46 & 9 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

$$
V(1, 1, 1) = \left(\frac{1 - t - t^2}{1 + t}, \frac{t - t^2 - t^3}{1 + t} \right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 0 & \cdots \\
7 & 4 & 1 & 0 & 0 & \cdots \\
31 & 18 & 6 & 1 & 0 & \cdots \\
154 & 90 & 33 & 8 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The first column of the matrix $U(1, 1, 1)$ corresponds to sequence A137644 in [16], and that of $V(1, 1, 1)$ corresponds to sequence A007863 in [16]. It follows from Theorem 4.1 that

$$U_{n,0} + U_{n,1} = (n + 1)V_{n,0}.$$

Example 5.2. If $b = 0, a = d = 1$, then

$$
U(1, 0, 1) = \left(\frac{1 - 3t^2 - 2t^3}{1 + t}, t - t^2 \right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
4 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\
13 & 7 & 3 & 1 & 0 & 0 & 0 & \cdots \\
46 & 24 & 11 & 4 & 1 & 0 & 0 & \cdots \\
166 & 86 & 40 & 16 & 5 & 1 & 0 & \cdots \\
610 & 314 & 148 & 62 & 22 & 6 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

$$
V(1, 0, 1) = \left(1 - t, t - t^2 \right)^{-1} = (C(t), tC(t)) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\
5 & 5 & 3 & 1 & 0 & 0 & 0 & \cdots \\
14 & 14 & 9 & 4 & 1 & 0 & 0 & \cdots \\
42 & 42 & 28 & 14 & 5 & 1 & 0 & \cdots \\
132 & 132 & 90 & 48 & 20 & 6 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The first column of the matrix $U(1, 0, 1)$ corresponds to sequence A026641 in [16], and that of $V(1, 0, 1)$ corresponds to sequences A000108 in [16]. By Theorem 4.1, we have

$$U_{n,0} + U_{n,1} = (n + 1)V_{n,0}.$$

The matrix $V(1, 0, 1)$ is the Catalan matrix of the first kind [21]. Therefore, we obtain a new interpretation for the Catalan numbers.

Corollary 5.1. *The Catalan number C_n is equal to the number of lattice paths from $(0, 0)$ to (n, n) using steps $(1, 0)$, $(1, 1)$ and $(0, 2)$ and staying on or below the line $y = x$.*

Example 5.3. If $a = b = 0, d = 1$, then

$$
U(0, 0, 1) = (1 - 3t^2, t(1 - t^2))^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 4 & 0 & 1 & 0 & 0 & 0 & \cdots \\
15 & 0 & 5 & 0 & 1 & 0 & 0 & \cdots \\
0 & 21 & 0 & 6 & 0 & 1 & 0 & \cdots \\
84 & 0 & 28 & 0 & 7 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

$$
V(0, 0, 1) = (1 - t^2, t(1 - t^2))^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & \cdots \\
3 & 0 & 3 & 0 & 1 & 0 & 0 & \cdots \\
0 & 7 & 0 & 4 & 0 & 1 & 0 & \cdots \\
12 & 0 & 12 & 0 & 5 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The first column of the matrix $U(0, 0, 1)$ corresponds to sequence A005809 in [16], and that of $V(0, 0, 1)$ corresponds to sequence A001764 in [16]. It follows from Theorem 4.1 that

$$U_{n,0} = (n + 1)V_{n,0}.$$

Example 5.4. If $a = 0, b = d = 1$, then

$$
U(0, 1, 1) = \left(1 - 2t - 3t^2, t - t^2 - t^3\right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
9 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\
40 & 14 & 4 & 1 & 0 & 0 & 0 & \cdots \\
190 & 65 & 20 & 5 & 1 & 0 & 0 & \cdots \\
924 & 315 & 98 & 27 & 6 & 1 & 0 & \cdots \\
4578 & 1554 & 490 & 140 & 35 & 7 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

$$
V(0, 1, 1) = \left(1 - t - t^2, t - t^2 - t^3\right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\
10 & 7 & 3 & 1 & 0 & 0 & 0 & \cdots \\
38 & 26 & 12 & 4 & 1 & 0 & 0 & \cdots \\
154 & 105 & 49 & 18 & 5 & 1 & 0 & \cdots \\
654 & 444 & 210 & 80 & 25 & 6 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The first column of the matrix $U(0, 1, 1)$ corresponds to sequence A038112 in [16], and that of $V(0, 1, 1)$ corresponds to sequence A001002 in [16]. It follows from Theorem 4.1 that

$$U_{n,0} = (n + 1)V_{n,0}.$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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