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Spectra of the Edge-Subdivision-Vertex and Edge-Subdivision-Edge Coronae

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Abstract In this paper two classes of corona are defined: edge-subdivision-vertex corona $G_1 \vee G_2$ and edge-subdivision-edge corona $G_1 \forall G_2$. Then, the $\boldsymbol{A}$-spectrum (respectively, $\boldsymbol{L}$-spectrum, $\boldsymbol{Q}$-spectrum) of the two classes of new graphs are given in terms of the corresponding spectra of G_1 and G_2. By using the Laplacian spectra, the number of spanning trees and Kirchhoff index of $G_1 \vee G_2$ and $G_1 \forall G_2$ are obtained.

Key words spectrum; edge-subdivision-vertex corona; edge-subdivision-edge corona; spanning tree; Kirchhoff index

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边剖分点冠图和边剖分边冠图的谱

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摘 要 定义两个图 G_1和 G_2的冠图:边剖分点冠图 $G_1 \vee G_2$和边剖分边冠图 $G_1 \forall G_2$;并用原图的邻接谱、拉普拉斯谱、无符号拉普拉斯谱表示两类冠图的邻接谱、拉普拉斯谱、无符号拉普拉斯谱. 基于拉普拉斯谱,给出并证明两类冠图 $G_1 \vee G_2$和 $G_1 \forall G_2$的生成树数目以及 Kirchhoff 指数.

关键词 谱; 边剖分点冠图; 边剖分边冠图; 生成树; Kirchhoff 指数

Let $G=(V,E)$ be a graph with vertex set $V=\{v_1, v_2, \cdots, v_n\}$ and edge set E. The adjacency matrix of G is denoted by $\boldsymbol{A}$. The Laplacian matrix of G is defined as $\boldsymbol{L}=\boldsymbol{D}-\boldsymbol{A}$. Denote the eigenvalues of $\boldsymbol{L}$ by $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$. The signless Laplacian matrix of G is defined as $\boldsymbol{Q}=\boldsymbol{D}+\boldsymbol{A}$. For a graph G, we call f_A (*respectively*, f_L, f_Q) the adjacent (respectively, Laplacian, signless Laplacian) characteristic polynomial of G [1-3]. Calculating the spectra of graphs as well as formulating the characteristic polynomials of graphs is a fundamental and very meaningful work in spectral graph theory.

The subdivision graph $S(G)$ [3] of a graph G is the graph obtained by inserting a new vertex into every edge of G. We denote the set of such new vertices by $I(G)$.

Definition 1 The edge-subdivision-vertex corona of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from G_1 and $|E(G_1)|$ copies of $S(G_2)$ with each edge of G_1 corresponding to one copy of

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$S(G_2)$ and all vertex-disjoint, by joining end-vertex of the i th edge of $E(G_1)$ to each vertex of $V(G_2)$ in the i th copy of $S(G_2)$.

Definition 2 The edge-subdivision-edge corona of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \forall G_2$, is the graph obtained from G_1 and $|E(G_1)|$ copies of $S(G_2)$ with each edge of G_1 corresponding to one copy of $S(G_2)$ and all vertex-disjoint, by joining end-vertex of the i th edge of $E(G_1)$ to each vertex of $I(G_2)$ in the i th copy of $S(G_2)$.

The paper is organized as follows. In section 1, some lemmas used in this paper are given. In section 2, the $\boldsymbol{A}$-spectrum (respectively, $\boldsymbol{L}$-spectrum, $\boldsymbol{Q}$-spectrum) of edge-subdivision-vertex corona $G_1 \vee G_2$ for an regular graph G_1 and an regular graph G_2 (see Theorems 1, 2, 3) are computed. In section 3, the $\boldsymbol{A}$-spectrum (respectively, $\boldsymbol{L}$-spectrum, $\boldsymbol{Q}$-spectrum) of edge-subdivision-edge corona $G_1 \forall G_2$ for an regular graph G_1 and an regular graph G_2 (see Theorems 4, 5, 6) are obtained. By Theorems 2 and 5, the number of spanning tree and Kirchhoff index of $G_1 \vee G_2$ and $G_1 \forall G_2$ (see Corollaries 2, 3, 5, 6) are obtained.

1 Some Lemmas

Lemma 1 [4-5] The $\boldsymbol{M}$-coronal $\Gamma_M(x)$ of a square matrix $\boldsymbol{M}$ is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is,

$$\Gamma_M(x) = \mathbf{1}_n^{\mathrm{T}}(x\boldsymbol{I}_n - \boldsymbol{M})^{-1}\mathbf{1}_n, \tag{1}$$

where $\mathbf{1}_n$ denotes the column vector of size n with all the entries equal 1.

Lemma 2 [6] Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3$ and $\boldsymbol{M}_4$ be respectively $p\times p, p\times q, q\times p$ and $q\times q$ matrices with $\boldsymbol{M}_1$ and $\boldsymbol{M}_4$ invertible. Then

$$\det\begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{pmatrix} = \det(\boldsymbol{M}_4)\cdot\det(\boldsymbol{M}_1 - \boldsymbol{M}_2\boldsymbol{M}_4^{-1}\boldsymbol{M}_3) \tag{2}$$

$$= \det(\boldsymbol{M}_1)\cdot\det(\boldsymbol{M}_4 - \boldsymbol{M}_3\boldsymbol{M}_1^{-1}\boldsymbol{M}_2), \tag{3}$$

where $\boldsymbol{M}_1 - \boldsymbol{M}_2\boldsymbol{M}_4^{-1}\boldsymbol{M}_3$ and $\boldsymbol{M}_4 - \boldsymbol{M}_3\boldsymbol{M}_1^{-1}\boldsymbol{M}_2$ are the Schur complements of $\boldsymbol{M}_4$ and $\boldsymbol{M}_1$, respectively.

Lemma 3 [7] The Kronecker product $\boldsymbol{A}\otimes\boldsymbol{B}$ of two matrices $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{p\times q}$ is the $mp\times nq$ matrix obtained from $\boldsymbol{A}$ by replacing each element a_{ij} by $a_{ij}\boldsymbol{B}$,

$$A\otimes B = (a_{ij}\boldsymbol{B})_{mp\times nq}. \tag{4}$$

Lemma 4 [2] Let $t(G)$ denote the number of spanning tree of G, it is well known that if G is a connected graph on n vertices with Laplacian spectrum $\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_{n-1}(G)>\mu_n(G)=0$, then

$$t(G) = \frac{\mu_1(G)\mu_2(G)\cdots\mu_{n-1}(G)}{n}. \tag{5}$$

The Kirchhoff index of a graph G_1, denoted by $kf(G)$, is defined as the sum of resistance distances between all pairs of vertices [8]. Gutman et al. [9] proved that the Kirchhoff index of a connected graph n_1 with $n(n\geqslant 2)$ vertices.

Lemma 5 [9] The Kirchhoff index of a connected graph G with $n(n\geqslant 2)$ vertices can be expressed as

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i(G)}. \tag{6}$$

2 Spectra of Edge-subdivision-vertex Corona

2.1 A-spectrum of edge-subdivision-vertex corona

Theorem 1 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$f_A = x^{m_1m_2-m_1n_2}\cdot(x^2-2r_2)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-\lambda_i(G_2)-r_2)^{m_1}\cdot\prod_{i=1}^{n_1}(x^3-\lambda_i(G_1)x^2-(2r_2+n_2r_1+\lambda_i(G_1)n_2)x+2r_2\lambda_i(G_1)).$$

Proof Let $\boldsymbol{A}_1$ denote the adjacency matrices of G_1. Then, with respect to the partition $V(G_1)\cup[U_1\cup U_2\cup\cdots\cup U_{n_2}]\cup[E_1\cup E_2\cup\cdots\cup E_{m_2}]$ of $V(G_1\vee G_2)$, the adjacency matrix of $G_1\vee G_2$ can be written as

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{1}_{n_2}^T\otimes\boldsymbol{R}_1 & \boldsymbol{0}_{n_1\times m_1m_2}\\ \boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^T & \boldsymbol{0}_{m_1n_2\times m_1n_2} & \boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1}\\ \boldsymbol{0}_{m_1m_2\times n_1} & \boldsymbol{R}_2^T\otimes\boldsymbol{I}_{m_1} & \boldsymbol{0}_{m_1m_2\times m_1m_2}\end{bmatrix},$$

where $\boldsymbol{0}_{s\times t}$ denotes the $s\times t$ matrix with all entries equal to zero, $\boldsymbol{I}_n$ is the identity matrix of order n, $\boldsymbol{1}_m$ denotes the column vector of size m with all the entries equal one. Thus the adjacency characteristic polynomial of $G_1\vee G_2$ is given by

$$f_A=\det\begin{bmatrix}x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1 & -\boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1 & \boldsymbol{0}_{n_1\times m_1m_2}\\ -\boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & x\boldsymbol{I}_{m_1n_2} & -\boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1}\\ \boldsymbol{0}_{m_1m_2\times n_1} & -\boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1} & x\boldsymbol{I}_{m_1m_2}\end{bmatrix}=x^{m_1m_2}\cdot\det(\boldsymbol{S}),$$

where

$$\boldsymbol{S}=\begin{pmatrix}x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1 & -\boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1\\ -\boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & x\boldsymbol{I}_{m_1n_2}\end{pmatrix}-\begin{pmatrix}\boldsymbol{0}_{n_1\times m_1m_2}\\ -\boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1}\end{pmatrix}(x\boldsymbol{I}_{m_1m_2})^{-1}(\boldsymbol{0}_{m_1m_2\times n_1}\ \ -\boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1})=$$

$$\begin{pmatrix}x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1 & -\boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1\\ -\boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & (x\boldsymbol{I}_{n_2}-\frac{1}{x}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}})\otimes\boldsymbol{I}_{m_1}\end{pmatrix}.$$

$$\det(\boldsymbol{S})=\det((x\boldsymbol{I}_{n_2}-\frac{1}{x}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}})\otimes\boldsymbol{I}_{m_1})\cdot\det(x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1-\Gamma_{\frac{1}{x}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}}}(x)\cdot\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}})=$$

$$x^{-m_1n_2}\cdot(x^2-2r_2)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-\lambda_i(G_2)-r_2)^{m_1}\cdot$$

$$\prod_{i=1}^{n_1}(x^3-\lambda_i(G_1)x^2-(2r_2+n_2r_1+\lambda_i(G_1)n_2)x+2r_2\lambda_i(G_1)).$$

2.2 *L*-spectrum of edge-subdivision-vertex corona

Theorem 2 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$f_L(x)=(x-2)^{m_1m_2-m_1n_2}\cdot(x^2-(4+r_2)x+4)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-(4+r_2)x+4+\mu_i(G_2))^{m_1}\cdot$$

$$\prod_{i=1}^{n_1}(x^3-(4+r_2+r_1n_2+\mu_i(G_1))x^2+(4+(2+r_2)n_2r_1+(4+r_2+n_2)\mu_i(G_1))x-(4+2n_2)\mu_i(G_1)).$$

Proof Let $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ be the incidence matrix of G_1 and G_2 respectively. Let $\boldsymbol{L}_1$ denote the Laplacian matrices of G_1. Then, the Laplacian matrix of $G_1\vee G_2$ can be written as

$$\boldsymbol{L}=\begin{bmatrix}\boldsymbol{L}_1+r_1n_2\boldsymbol{I}_{n_1} & -\boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1 & \boldsymbol{0}_{n_1\times m_1m_2}\\ -\boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & (r_2+2)\boldsymbol{I}_{n_2}\otimes\boldsymbol{I}_{m_1} & -\boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1}\\ \boldsymbol{0}_{m_1m_2\times n_1} & -\boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1} & 2\boldsymbol{I}_{m_1m_2}\end{bmatrix}.$$

Thus the Laplacian characteristic polynomial of $G_1\vee G_2$ is given by

$$f_L(x)=\det\begin{bmatrix}(x-r_1n_2)\boldsymbol{I}_{n_1}-\boldsymbol{L}_1 & \boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1 & \boldsymbol{0}_{n_1\times m_1m_2}\\ \boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & (x-r_2-2)\boldsymbol{I}_{n_2}\otimes\boldsymbol{I}_{m_1} & \boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1}\\ \boldsymbol{0}_{m_1m_2\times n_1} & \boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1} & (x-2)\boldsymbol{I}_{m_1m_2}\end{bmatrix}=(x-2)^{m_1m_2}\det(\boldsymbol{S}),$$

where

$$\boldsymbol{S}=\begin{bmatrix}(x-r_1n_2)\boldsymbol{I}_{n_1}-\boldsymbol{L}_1 & \boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1\\ \boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & ((x-2-r_2)\boldsymbol{I}_{n_2}-\frac{1}{x-2}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}})\otimes\boldsymbol{I}_{m_1}\end{bmatrix}$$

is the Schur complement [7] of $(x-2)\boldsymbol{I}_{m_1m_2}$. For any graph G, it is well known that $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}=\boldsymbol{A}+\boldsymbol{D}$ and $\boldsymbol{L}=\boldsymbol{D}-\boldsymbol{A}$, and $\boldsymbol{D}$ is the diagonal matrix. If G is an r-regular graph on n vertices, we have $\boldsymbol{D}=r\boldsymbol{I}_n$. Then $\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}}=\boldsymbol{A}_1+r_1\boldsymbol{I}_{n_1}=(\boldsymbol{D}_1-\boldsymbol{L}_1)+r_1\boldsymbol{I}_{n_1}=2r_1\boldsymbol{I}_{n_1}-\boldsymbol{L}_1$ and $\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}}=\boldsymbol{A}_2+r_2\boldsymbol{I}_{n_2}=(\boldsymbol{D}_2-\boldsymbol{L}_2)+r_2\boldsymbol{I}_{n_2}=2r_2\boldsymbol{I}_{n_2}-\boldsymbol{L}_2$. Thus, the result follows from

$$\det(S)=(\det((x-2-r_2)\boldsymbol{I}_{n_2}-\frac{\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}}}{x-2}))^{m_1}\det((x-r_1n_2)\boldsymbol{I}_{n_1}-\boldsymbol{L}_1-\Gamma_{\frac{\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}}}{x-2}}(x-2-r_2)\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}})=$$

$$(\det((x-2-r_2)\boldsymbol{I}_{n_2}-\frac{\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}}}{x-2}))^{m_1}\det((x-r_1n_2)\boldsymbol{I}_{n_1}-\boldsymbol{L}_1-\frac{n_2(x-2)}{(x-2)^2-r_2x}\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}})=$$

$$(x-2)^{-m_1m_2}\cdot(x-(r_2+4)x+4)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-(4+r_2)x+4+\mu_i(G_2))^{m_1}\cdot$$

$$\prod_{i=1}^{n_1}(x^3-(4+r_2+r_1n_2+\mu_i(G_1))x^2+(4+(2+r_2)n_2r_1+(4+r_2+n_2+\mu_i(G_1))x-$$

$$(4+2n_2)\mu_i(G_1)).$$

Corollary 1 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then the Laplacian spectrum is

(a) 2, repeated $m_1m_2-m_1n_2$ times;

(b) Two roots of the equation $x^2-(4+r_2)x+4=0$, for each root repeated m_1-n_1 times;

(c) Two roots of the equation $x^2-(4+r_2)x+2r_2+\mu_i(G_2)=0$, for each root repeated m_1 times, for $i=2,\cdots,n_2$;

(d) Three roots of the equation

$$x^3-(4+r_2+r_1n_2+\mu_i(G_1))x^2+(4+(2+r_2)n_2r_1+(4+r_2+n_2)\mu_i(G_1))x-(4+2n_2)\mu_i(G_1)=0,$$

for $i=2,\cdots,n_1$.

Corollary 2 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$t(G_1\vee G_2)=\frac{2^{m_1m_2+2m_1-m_1n_2-2n_1}(4+2n_2r_1+n_2r_1r_2)t(G_1)}{m_1m_2+m_1n_2+n_1}\cdot(4+2n_2)^{n-1}\cdot\prod_{i=2}^{n_2}(4+\mu_i(G_2))^{m_1}.$$

Corollary 3 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$Kf(G_1\vee G_2)=\Big(m_1m_2+m_1n_2+n_1\Big)\times\Big(\frac{m_1m_2-m_1n_2}{2}+\frac{(4+r_2)(m_1-n_1)}{4}+\frac{4+r_2+r_1n_2}{4+2n_2r_1+n_2r_1r_2}+$$

$$\frac{(4+r_2+n_2)(n_1-1)}{4+2n_2}+\frac{4+(2+r_2)n_2r_1}{(4+n_2)n_1}\cdot Kf(G_1)+\sum_{i=2}^{n_2}\frac{(4+r_2)m_1}{4+\mu_i(G_2)}\Big).$$

2.3 Q-spectrum of edge-subdivision-vertex corona

Theorem 3 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$f_Q = (x-2)^{m_1m_2-m_1n_2} \cdot (x^2-(4+r_2)x+4)^{m_1-n_1} \cdot \prod_{i=1}^{n_2-1}(x^2-(4+r_2)x+4+2r_2-\nu_i(G_2))^{m_1} \cdot \prod_{i=1}^{n_1}(x^3-(4+r_2+r_1n_2+\nu_i(G_1))x^2+(4+(4+r_2)n_2r_1+(4+r_2-n_2)\nu_i(G_1))x+(2n_2-4)\nu_i(G_1)-4r_1n_2).$$

Proof The proof is similar as Theorem 2 and is omitted.

3 Spectra of Edge-subdivision-edge Corona

3.1 A-spectrum of edge-subdivision-edge corona

Theorem 4 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$f_A = x^{n_1m_2-n_1n_2-n_1} \cdot \prod_{i=1}^{n_1}(x^4-\lambda_i(G_1)x^3-(m_2+2r_2)x^2+2r_2\lambda_i(G_1)x+2r_2m_2-n_2r_2^2) \cdot \prod_{i=2}^{n_2}(x^2-\lambda_i(G_2)-r_2)^{n_1}.$$

Proof The adjacency matrix of $G_1 \forall G_2$ can be written as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{0}_{n_1\times m_1n_2} & \boldsymbol{1}_{m_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1 \\ \boldsymbol{0}_{m_1n_2\times n_1} & \boldsymbol{0}_{m_1n_2\times m_1n_2} & \boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1} \\ \boldsymbol{1}_{m_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & \boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1} & \boldsymbol{0}_{m_1m_2\times m_1m_2} \end{bmatrix}.$$

Thus the adjacency characteristic polynomial of $G_1 \forall G_2$ is given by

$$f_A = \det\begin{bmatrix} x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1 & \boldsymbol{0}_{n_1m_1\times n_2} & -\boldsymbol{1}_{m_1}^{\mathrm{T}}\otimes\boldsymbol{R}_1 \\ \boldsymbol{0}_{m_1n_2\times n_1} & x\boldsymbol{I}_{m_1n_2} & -\boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1} \\ -\boldsymbol{1}_{m_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & -\boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1} & x\boldsymbol{I}_{m_1m_2} \end{bmatrix} = x^{m_1m_2}\cdot\det(S),$$

where

$$\boldsymbol{S} = \begin{pmatrix} x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1 & \boldsymbol{0}_{n_1m_1\times n_2} \\ \boldsymbol{0}_{m_1n_2\times n_1} & x\boldsymbol{I}_{m_1n_2} \end{pmatrix} - \begin{pmatrix} -\boldsymbol{1}_{m_1}^{\mathrm{T}}\otimes\boldsymbol{R}_1 \\ -\boldsymbol{R}_2\otimes\boldsymbol{I}_{m_1} \end{pmatrix}(x\boldsymbol{I}_{m_1m_2})^{-1}(-\boldsymbol{1}_{m_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} - \boldsymbol{R}_2^{\mathrm{T}}\otimes\boldsymbol{I}_{m_1}) = \begin{pmatrix} x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1-\frac{m_2}{x}\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}} & -\frac{r_2}{x}\boldsymbol{1}_{n_2}^{\mathrm{T}}\otimes\boldsymbol{R}_1 \\ -\frac{r_2}{x}\boldsymbol{1}_{n_2}\otimes\boldsymbol{R}_1^{\mathrm{T}} & (x\boldsymbol{I}_{n_2}-\frac{1}{x}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}})\otimes\boldsymbol{I}_{m_1} \end{pmatrix}.$$

$$\det(S) = \det((x\boldsymbol{I}_{n_2}-\frac{1}{x}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}})\otimes\boldsymbol{I}_{m_1}) \cdot \det(x\boldsymbol{I}_{n_1}-\boldsymbol{A}_1-\frac{m_2}{x}\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}}-\frac{r_2^2}{x^2}\Gamma_{\frac{1}{x}\boldsymbol{R}_2\boldsymbol{R}_2^{\mathrm{T}}}(x)\boldsymbol{R}_1\boldsymbol{R}_1^{\mathrm{T}}) = x^{-m_1n_2-n_1}\cdot(x^2-2r_2)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-\lambda_i(G_2)-r_2)^{m_1}\cdot \prod_{i=1}^{n_1}(x^4-\lambda_i(G_1)x^3-(2r_2+\lambda_i(G_1)m_2+m_2r_1)x^2+2r_2\lambda_i(G_1)x+2\lambda_i(G_1)r_2m_2+2r_1r_2m_2-r_2^2n_2\lambda_i(G_1)-r_1r_2^2n_2).$$

3.2 L-spectrum of edge-subdivision-edge corona

Theorem 5 Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$f_L(x) = (x-4)^{m_1m_2-m_1n_2-n_1}(x^2-4x-r_2x+2r_2)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-4x-r_2x+2r_2+\mu_i(G_2))^{m_1}\cdot$$

$$\prod_{i=1}^{n_1}(x^4-(8+r_2+r_1m_2+\mu_i(G_1))x^3+(16+6r_2+(6+r_2)r_1m_2+(8+r_2+m_2)\mu_i(G_1))x^2-(8r_2+(8+4r_2)r_1m_2+(16+6r_2+4m_2+r_2m_2)\mu_i(G_1))x+(8r_2+2r_2m_2+n_2{r_2}^2)\mu_i(G_1)+(4m_2-2n_2r_2)r_1r_2).$$

Proof　The Laplacian matrix of $G_1 \forall G_2$ can be written as

$$L=\begin{bmatrix} L_1+r_1m_2I_{n_1} & 0_{n_1\times m_1n_2} & -1_{m_2}^{\mathrm{T}}\otimes R_1 \\ 0_{m_1n_2\times n_1} & r_2I_{n_2}\otimes I_{m_1} & -R_2\otimes I_{m_1} \\ -1_{m_2}\otimes R_1^{\mathrm{T}} & -R_2^{\mathrm{T}}\otimes I_{m_1} & 4I_{m_1m_2}\end{bmatrix}.$$

Thus the Laplacian characteristic polynomial of $G_1 \forall G_2$ is given by

$$f_L(x)=\det\begin{bmatrix}(x-r_1m_2)I_{n_1}-L_1 & 0_{n_1\times m_1n_2} & 1_{m_2}^{\mathrm{T}}\otimes R_1 \\ 0_{m_1n_2\times n_1} & (x-r_2)I_{m_1n_2} & R_2\otimes I_{m_1} \\ 1_{m_2}\otimes R_1^{\mathrm{T}} & R_2^{\mathrm{T}}\otimes I_{m_1} & (x-4)I_{m_1m_2}\end{bmatrix}=\det(S)\cdot(x-4)^{m_1m_2},$$

where

$$S=\begin{bmatrix}(x-r_1m_2)I_{n_1}-\frac{m_2}{x-4}R_1R_1^{\mathrm{T}}-L_1 & -\frac{r_2}{x-4}1_{n_2}^{\mathrm{T}}\otimes R_1 \\ -\frac{r_2}{x-4}1_{n_2}\otimes R_1^{\mathrm{T}} & ((x-r_2)I_{n_2}-\frac{1}{x-4}R_2R_2^{\mathrm{T}})\otimes I_{m_1}\end{bmatrix}.$$

$$\det(S)=(\det((x-r_2)I_{n_1}-\frac{1}{x-4}R_2R_2^{\mathrm{T}}))^{m_1}\cdot\det((x-r_1m_2)I_{n_1}-L_1-\frac{m_2}{x-4}R_1R_1^{\mathrm{T}}-\frac{r_2^2}{(x-4)^2}\Gamma_{\frac{R_2R_2^{\mathrm{T}}}{x-4}}(x-r_2)R_1R_1^{\mathrm{T}})=$$

$$(x-4)^{-m_1n_2-n_1}(x^2-4x-r_2x+2r_2)^{m_1-n_1}\cdot\prod_{i=2}^{n_2}(x^2-4x-r_2x+2r_2+\mu_i(G_2))^{m_1}\cdot\prod_{i=1}^{n_1}(x^4-(8+r_2+r_1m_2+\mu_i(G_1))x^3+(16+6r_2+(6+r_2)r_1m_2+(8+r_2+m_2)\mu_i(G_1))x^2-(8r_2+(8+4r_2)r_1m_2+(16+6r_2+4m_2+r_2m_2)\mu_i(G_1))x+(8r_2+2r_2m_2+n_2r_2^2)\mu_i(G_1)+(4m_2-2n_2r_2)r_1r_2).$$

Corollary 4　Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then the Laplacian spectrum is

(a) 4, repeated $m_1m_2-m_1n_2-n_1$ times;

(b) Two roots of the equation $x^2-(4+r_2)x+2r_2=0$, for each root repeated m_1-n_1 times;

(c) Two roots of the equation $x^2-(4+r_2)x+2r_2+\mu_i(G_2)=0$, for each root repeated $m_1m_1m_1$ times, for $i=2,\cdots,n_2$;

(d) Four roots of the equation

$$x^4-(8+r_2+r_1m_2+\mu_i(G_1))x^3+(16+6r_2+(6+r_2)r_1m_2+(8+r_2+m_2)\mu_i(G_1))x^2-(8r_2+(8+4r_2)r_1m_2+(16+6r_2+4m_2+r_2m_2)\mu_i(G_1))x+(8r_2+2r_2m_2+n_2r_2^2)\mu_i(G_1)+(4m_2-2n_2r_2)r_1r_2=0 \text{ for } i=2,\cdots,n_1.$$

Corollary 5　Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$t(G_1\forall G_2)=\frac{4^{m_1m_2-m_1n_2-n_1}(2r_2)^{m_1-n_1}}{n_1+m_1m_2+m_1n_2}\cdot\prod_{i=2}^{n_2}(2r_2+\mu_i(G_2))^{m_1}\cdot\prod_{i=1}^{n_1}((8r_2+2r_2m_2+n_2r_2^2)\mu_i(G_1)+(4m_2-2n_2r_2)r_1r_2);$$

$$kf(G_1 \forall G_2) = (m_1m_2 + m_1n_2 + n_1) \times \left(\frac{m_1m_2 - m_1n_2 - n_1}{4} + \frac{(m_1 - n_1)(4 + r_2)}{2r_2} + \sum_{i=2}^{n_2} \frac{(4 + r_2)m_1}{2r_2 + \mu_i(G_2)} + \right.$$
$$\left. \sum_{i=1}^{n_1} \frac{8r_2 + (8 + 4r_2)r_1m_2 + (16 + 6r_2 + 4m_2 + m_2r_2)\mu_i(G_1)}{(4m_2 - 2n_2r_2)r_1r_2 + (8r_2 + 2m_2r_2 + n_2r_2^2)\mu_i(G_1)}\right).$$

3.3 Q-spectrum of edge-subdivision-edge corona

Theorem 6 Let G_1 be an r_1-regular graph on n_1 vertices, and G_2 an r_2-regular graph on n_2 vertices and m_2 edges. Then

$$f_Q = (x - 4)^{m_1m_2 - m_1n_2 - n_1} \cdot (x^2 - 4x - r_2x + 2r_2)^{m_1 - n_1} \cdot \prod_{i=1}^{n_2-1} (x^2 - 4x - r_2x + 4r_2 - \nu_i(G_2))^{m_1}$$
$$\prod_{i=1}^{n_1} ((x^4 - (8 + r_2 + r_1m_2 + \nu_i(G_1))x^3 + (16 + 6r_2 + (8 + r_2)r_1m_2 + (8 + r_2 - m_2)\nu_i(G_1))x^2 -$$
$$(8r_2 + (16 + 6r_2)r_1m_2 + (16 + 6r_2 - 4m_2 - r_2m_2)\nu_i(G_1))x +$$
$$(8r_2 + n_2r_2^2 - 2m_2r_2)\nu_i(G_1) + 8r_1r_2m_2)$$

Proof The proof is similar as Theorem 5 and is omitted.

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