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ORIGINAL ARTICLE

A bisection method for the milling of NURBS mapping projection curves by CNC machines

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Abstract The concept of non-uniform rational B-spline (NURBS) mapping projection curves (NURBS-MPCs) is proposed in this work. A NURBS-MPC is a projection curve on a NURBS surface of a NURBS curve. The bisection method is used to interpolate NURBS-MPCs. Using an assigned chord, the interpolation of a NURBS-MPC can be obtained easily, and the milling precision of the NURBS-MPC can be controlled effectively. Based on the NURBS theory, the bisection method, and the parametric programming method, an online NURBS software package (NURBS-SP) for FANUC 0i-MB/MC/MD CNC system and an offline NURBS toolbox (NURBS-T) for Matlab have been developed. Using an example of a planar NURBS curve, a NURBS-MPC is created on a NURBS surface. The simulation and milling of the NURBS-MPC show that the bisection method is feasible and effective. The online NURBS-SP endows the NURBS interpolation function for those CNC systems only equipped with linear interpolation (G01) and circular interpolation (G02/G03) and extends the interpolation functions and machining capability of low-middle level three-axis milling machines. The interactive application of the NURBS-T and NURBS-SP can accomplish the design, simulation, and milling for NURBS-MPCs. This feature makes them to have broad application prospects in CNC machining industry.

Keywords NURBS mapping projection curves · NURBS surface · Bisection method · NURBS toolbox · NURBS software package

1 Introduction

In the research of complex curves and surfaces, the non-uniform rational B-spline (NURBS) method is widely used in computer-aided geometric design (CAGD) and computer-aided design (CAD) due to its generality in the expression of curves and surfaces [1, 2]. By manipulating the control points as well as weights, the NURBS method provides the flexibility to design a largely variety of shapes [3]. That is the reason why the international organization for standardization (ISO) promulgated that the NURBS method is the unique mathematical method for defining industrial product shapes in STEP.

In the community of CNC machines and technologies, how to realize the machining of NURBS curves and surfaces has been always a research hotspot using various processes, such as five-axis CNC milling [4, 5], high-energy micro-arc spark—computer numerical control deposition (HEMAS-CNCD) [6, 7], and WEDM [8].

Unfortunately, low-middle level three-axis CNC systems (such as FANUC, Siemens, HNC, etc.) are not generally equipped with the NURBS interpolation function. This results in that programmers have to resort to some computer-aided manufacturing (CAM) softwares to accomplish the

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programming of NURBS curves and surfaces for low-middle level three-axis CNC milling. So it is an urgent problem to be solved and an exciting thing to develop NURBS curves/surfaces interpolation function for such level three-axis CNC milling machines.

Various interpolation methods for NURBS curves and surfaces have been investigated in the literature [9–11]. How to realize interpolation for a projection curve on a NURBS surface of a NURBS curve has never been focused on up to present. In order to accomplish this task, the bisection method is employed to realize the interpolation of NURBS mapping projection curves (MPCs). Additionally, an offline NURBS toolbox (NURBS-T) for Matlab and an online NURBS software package (NURBS-SP) for FANUC 0i-MB/MC/MD CNC system have been developed.

2 NURBS curve representation

In modern CAD/CAM systems, parts with complex curves and (or) surfaces (e.g., aircraft models, blades, and module dies) are generally written in parametric curves/surfaces [6] which can provide designers or CNC programmers with high machining quality, precision, and efficiency [12–15]. In parametric curves, NURBS curve and surface are becoming more and more important and widely used in the CAD/CAM community [13, 16–18].

2.1 NURBS curve representation

A NURBS curve is represented parametrically as follows:

$$\boldsymbol{C}(u)=\frac{\sum_{i=0}^{n} N_{i,p}(u)w_i\boldsymbol{P}_i}{\sum_{i=0}^{n} N_{i,p}(u)w_i}\quad (a\leq u\leq b) \tag{1}$$

where $\{\boldsymbol{P}_i\}$ are control points, $\{w_i\}$ are the weights, p is the degree of the NURBS curve, u is the knot value ($u\in\boldsymbol{U}$), and $\boldsymbol{U}$ is the knot vector as follows:

$$\boldsymbol{U}=\left\{\underbrace{a,\cdots,a,}_{p+1}u_{p+1},\cdots,u_{m-p-1},\underbrace{b,\cdots,b}_{p+1}\right\}(m=n+p+1) \tag{2}$$

$\{N_{i,p}(u)\}$ is called the pth degree basis function. Recursive formulae for the basis function $N_{i,p}(u)$ can be expressed as follows:

$$\begin{cases} N_{i,0}(u)=\begin{cases}1 & for \quad u_i\leq u<u_{i+1}\\ 0 & \text{otherwise}\end{cases}\\ N_{i,p}(u)=\dfrac{u-u_i}{u_{i+p}-u_i}N_{i,p\text{-}1}(u)+\dfrac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}}N_{i+1,p\text{-}1}(u)\\ \text{defined}\,\dfrac{0}{0}=0\end{cases} \tag{3}$$

Additionally, a rational basis function $R_{i,p}(u)$ can be defined as follows:

$$R_{i,p}(u)=\frac{N_{i,p}(u)w_i}{\sum_{j=0}^{n} N_{j,p}(u)w_j}\quad (0\leq u\leq 1) \tag{4}$$

As a result, Eq. (1) can be rewritten as follows:

$$\boldsymbol{C}(u)=\sum_{i=0}^{n} R_{i,p}(u)\boldsymbol{P}_i\quad (0\leq u\leq 1) \tag{5}$$

2.2 NURBS surface representation

A NURBS surface is a bivariate vector-valued piecewise rational function. The form of a NURBS surface is presented as follows:

$$\boldsymbol{S}(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}\boldsymbol{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}}\quad (0\leq u,v\leq 1) \tag{6}$$

where $\{\boldsymbol{P}_{i,j}\}$ form a bidirectional control point net, $\{w_{i,j}\}$ are the weights, and $\{N_{i,p}(u)\}$ and $\{N_{j,q}(u)\}$ are the nonrational B-spline basis functions defined on $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. The knot vectors of $\boldsymbol{U}$ and $\boldsymbol{V}$ are presented as follows:

$$\boldsymbol{U}=\left\{\underbrace{0,\cdots,0,}_{p+1}u_{p+1},\cdots,u_{r-p\text{-}1},\underbrace{1,\cdots,1}_{p+1}\right\}(r=n+p+1) \tag{7}$$

$$\boldsymbol{V}=\left\{\underbrace{0,\cdots,0,}_{q+1}u_{q+1},\cdots,u_{s-q-1},\underbrace{1,\cdots,1}_{q+1}\right\}(s=m+q+1) \tag{8}$$

Furthermore, the piecewise rational basis function of $\{R_{i,j}(u,v)\}$ can be defined as follows:

$$R_{i,j}(u,v)=\frac{N_{i,p}(u)N_{j,q}(v)w_{i,j}}{\sum_{k=0}^{n}\sum_{l=0}^{\mathrm{m}} N_{k,p}(u)N_{l,q}(v)w_{k,l}}\quad (0\leq u,v\leq 1) \tag{9}$$

As a result, Eq. (6) is rewritten as follows:

$$\boldsymbol{S}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}(u,v)\boldsymbol{P}_{i,j} \tag{10}$$

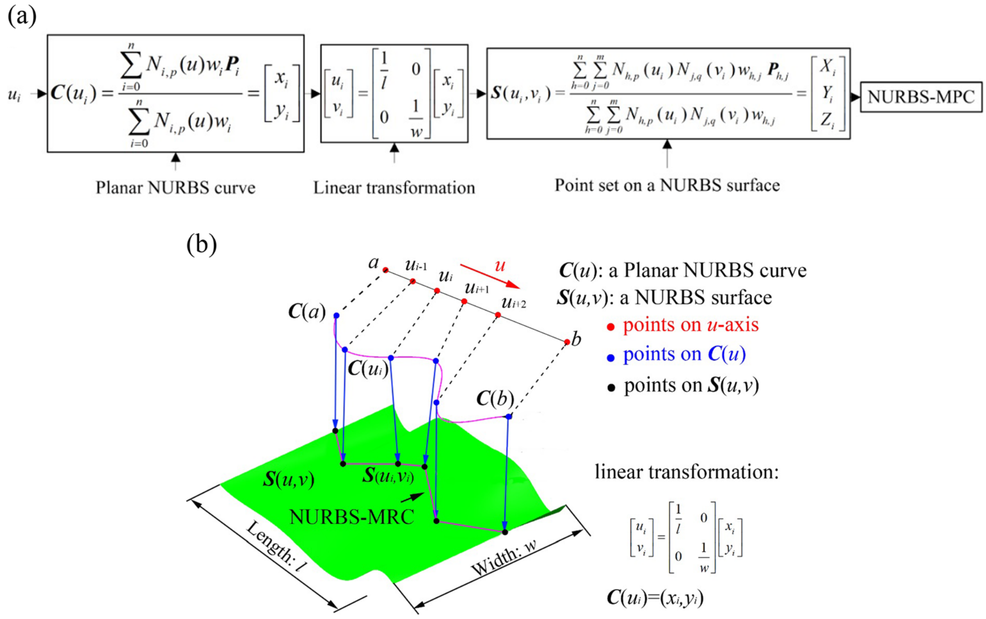


Fig. 1 A NURBS mapping projection curve on a NURBS surface. **a** The block diagram of algorithm. **b** The schematic image of NURBS mapping projection curve

3 The concept of NURBS mapping projection curves

A planar NURBS curve $\boldsymbol{C}(u)$ can be designed through assigning proper values to $\{\boldsymbol{P}_i\}$, $\{w_i\}$, $\boldsymbol{U}$, and p as depicted in Fig. 1. When the fixed step interpolation (FSI) is used, the u_{i+1} is determined as follows:

$$u_{i+1} = u_i + \Delta \tag{11}$$

where Δ is the fixed step.

The coordinate of the point on $\boldsymbol{C}(u)$ corresponding to $u_{\mathrm{i+1}}$ can be obtained as follows:

$$\left(x_{i+1}, y_{i+1}\right) = \boldsymbol{C}(u_{i+1}) = \boldsymbol{C}(u_i + \Delta) \tag{12}$$

when x and y in Eq. (12) are used as variables of $\boldsymbol{S}(u,v)$, a spatial point set on the NURBS surface of $\boldsymbol{S}(u,v)$ can be obtained. Theoretically, the point set creates a spatial curve on $\boldsymbol{S}(u,v)$ under the condition that the value of the fixed step (Δ) is assigned to be infinitely small. The spatial curve can be defined as the NURBS-MPC of the planar NURBS curve.

For low-middle level three-axis milling machines, they are equipped with only linear interpolation (G01) and circular interpolation (G02/G03). Unfortunately, G02/G03 can only be used in the three basic planes (XOY, YOZ, and XOZ), which makes it impossible to realize circular interpolation for random spatial points using G02/G03. In the milling of the NURBS-MPC abovementioned, the unique choice of interpolations is linear interpolation for the reason that the NURBS mapping projection points on the NURBS surface are spatial points but not planar points generally.

4 The bisection method for NURBS mapping projection curves

For the three-axis CNC milling process of a NURBS-MPC, an important question is how to realize its interpolation under the condition of meeting precision requirements. Apparently, the FSI method is not qualified for this work [6].

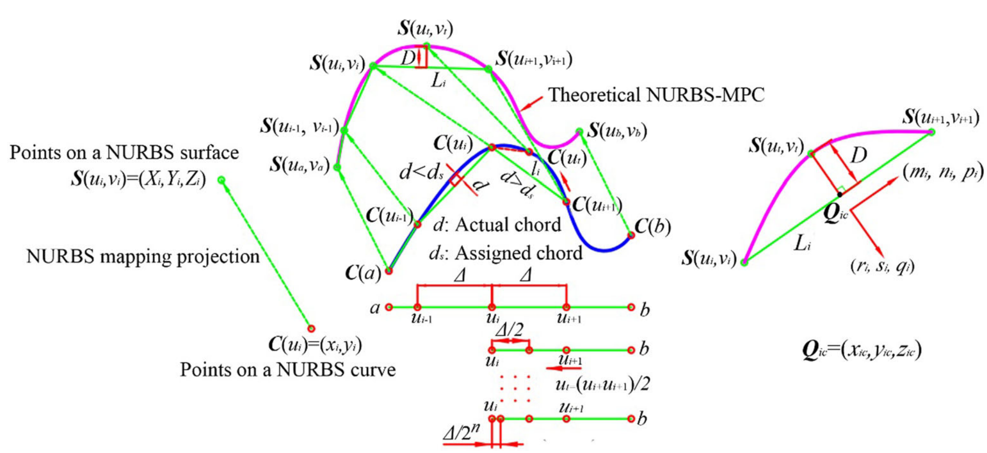


Fig. 2 The schematic of the bisection method for NURBS-MPC

As observed in Fig. 2, theoretically speaking, when the fixed step of Δ is assigned to be a sufficiently small value, the number of mapping projection points on a NURBS surface can be large enough when the FSI method is used. Then the length of a straight line between two mapping projection points can be very small. This makes the tiny line sequence approach a theoretical NURBS-MPC. However, too many tiny lines are not necessary when the precision requirements are met for the reason that it can increase the computational burden and decrease the efficiency of milling.

Assuming that the knot values of the interval $[u_i\ u_{i+1}]$ are u_i and u_{i+1}, respectively (as shown in Fig. 2), the points on the planar NURBS curve of $\boldsymbol{C}(u)$ are $\boldsymbol{C}(u_i)$ and $\boldsymbol{C}(u_{i+1})$. The midpoint of u_i and u_{i+1} on u-axis is $u_t = (u_i + u_{i+1})/2$ which corresponds to the point of $\boldsymbol{C}(u_t)$. The line segment determined by $\boldsymbol{C}(u_i)$ and $\boldsymbol{C}(u_{i+1})$ can be expressed as follows:

$$\begin{aligned} &Ix + Jy + K = 0 \\ &I = \frac{1}{\boldsymbol{C}_x(u_{i+1}) - \boldsymbol{C}_x(u_i)} \\ &J = -\frac{1}{\boldsymbol{C}_y(u_{i+1}) - \boldsymbol{C}_y(u_i)} \\ &K = -\frac{\boldsymbol{C}_x(u_i)}{\boldsymbol{C}_x(u_{i+1}) - \boldsymbol{C}_x(u_i)} + \frac{\boldsymbol{C}_y(u_i)}{\boldsymbol{C}_y(u_{i+1}) - \boldsymbol{C}_y(u_i)} \end{aligned} \tag{13}$$

The actual chord (d, as shown in Fig. 2) can be expressed as follows:

$$d = \frac{\left| I \times \boldsymbol{C}_x(u_t) + J \times \boldsymbol{C}_y(u_t) + K \right|}{\sqrt{I^2 + J^2}} \tag{14}$$

When the points of $\boldsymbol{C}(u_i)$, $\boldsymbol{C}(u_{i+1})$, and $\boldsymbol{C}(u_t)$ are mapped on to a NURBS surface, their NURBS mapping projection points are $\boldsymbol{S}(u_i,v_i)$, $\boldsymbol{S}(u_{i+1},v_{i+1})$, and $\boldsymbol{S}(u_t,v_t)$, respectively, which are spatial points. The tiny line between $\boldsymbol{S}(u_i,v_i)$ and $\boldsymbol{S}(u_{i+1},v_{i+1})$ is an approaching line. As can be observed in Fig. 2, the parameter D is the distance between the point $\boldsymbol{S}(u_t,v_t)$ and the straight line segment determined by points of $\boldsymbol{S}(u_i,v_i)$ and $\boldsymbol{S}(u_{i+1},v_{i+1})$. The parameter D means the approximation degree of the tiny line to the theoretical NURBS-MPC and also denotes the machining precision when three-axis CNC milling process is carried out. The smaller the parameter D is, the higher the approximation degree is, and the higher the machining precision is.

For a spatial line of L_i (as shown in Fig. 2), its parameter equation can be written as follows:

$$\begin{cases} x = m_i t + \boldsymbol{S}_x(u_i, v_i) \\ y = n_i t + \boldsymbol{S}_y(u_i, v_i) \\ z = p_i t + \boldsymbol{S}_z(u_i, v_i) \end{cases} \tag{15}$$

The direction vector (m_i, n_i, p_i) of the line L_i can be calculated as follows:

$$\begin{bmatrix} m_i \\ n_i \\ p_i \end{bmatrix} = \boldsymbol{S}(u_{i+1}, v_{i+1}) - \boldsymbol{S}(u_i, v_i) \tag{16}$$

Assuming that the coordinate of a perpendicular point generated by the line L_i and its perpendicular line passing the

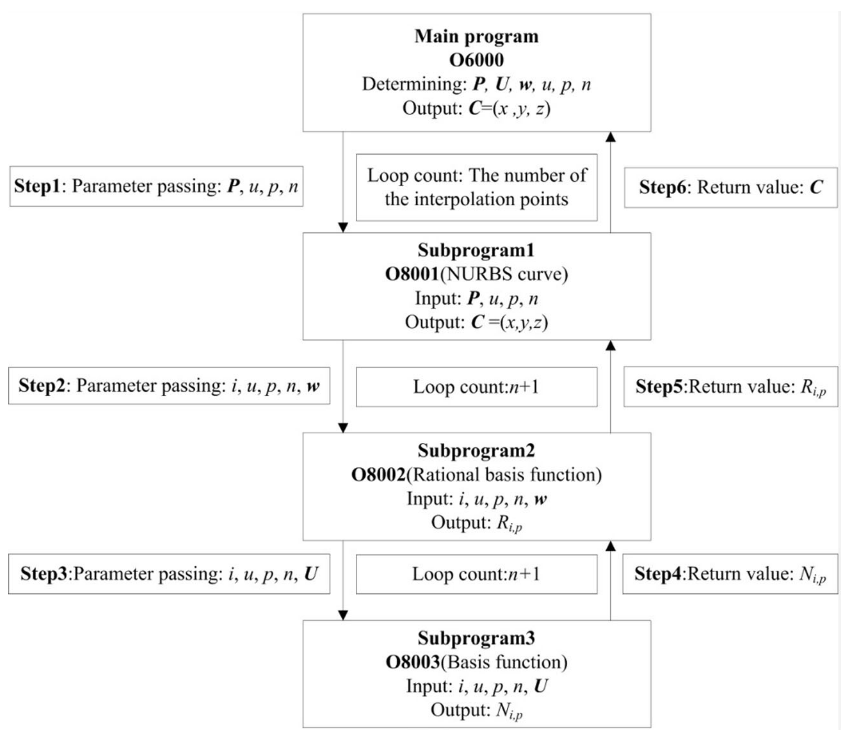


Fig. 3 The architecture system of the online NURBS-SP for FANUC 0i-MB/MC/MD

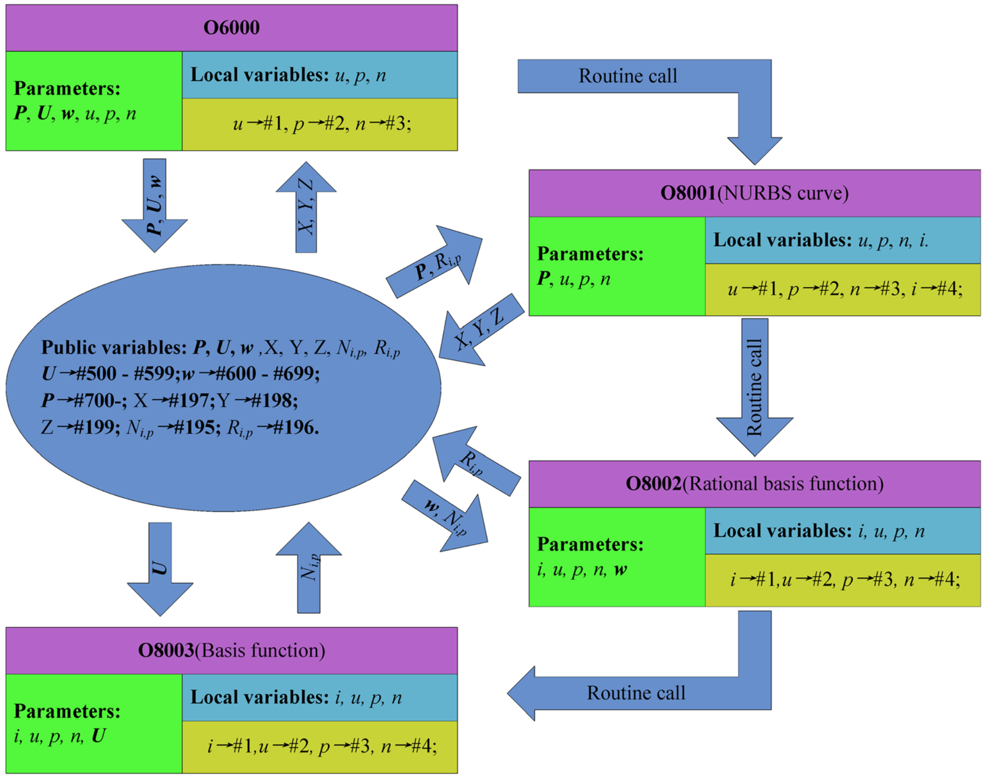


Fig. 4 The assignment of variables in NURBS-SP based on FANUC 0i-MB/MC/MD

point of $\boldsymbol{S}(u_t,v_t)$ is $\boldsymbol{Q}_{ic}$ (x_{ic}, y_{ic}, z_{ic}) as shown in Fig.2, the direction vector (r_i, s_i, q_i) of the perpendicular line can be expressed as follows:

$$\begin{bmatrix} r_i \\ s_i \\ q_i \end{bmatrix} = \boldsymbol{Q}_{ic} - \boldsymbol{S}(u_t, v_t) \tag{17}$$

Due to $(m_i, n_i, p_i)\cdot(r_i, s_i, q_i) = 0$ so the parameter of t_i for $\boldsymbol{Q}_{ic}$ in Eq. (15) is written as follows:

$$t_i = \frac{A+B+C}{{m_i}^2 + {n_i}^2 + {p_i}^2}$$
$$\begin{cases} A = m_i[\boldsymbol{S}_x(u_t, v_t) - \boldsymbol{S}_x(u_i, v_i)] \\ B = n_i[\boldsymbol{S}_y(u_t, v_t) - \boldsymbol{S}_y(u_i, v_i)] \\ C = p_i[\boldsymbol{S}_z(u_t, v_t) - \boldsymbol{S}_z(u_i, v_i)] \end{cases} \tag{18}$$

So, the coordinate of the perpendicular point can be counted as follows:

$$\boldsymbol{Q}_{ic} = \begin{bmatrix} x_{ic} \\ y_{ic} \\ z_{ic} \end{bmatrix} = \begin{bmatrix} m_i \\ n_i \\ p_i \end{bmatrix} t_i + \begin{bmatrix} \boldsymbol{S}_x(u_i, v_i) \\ S_y(u_i, v_i) \\ S_z(u_i, v_i) \end{bmatrix} \tag{19}$$

As a result, the parameter D can be calculated as follows:

$$\begin{aligned} D &= |\boldsymbol{S}(u_t, v_t) - \boldsymbol{Q}_{ic}| \\ &= \sqrt{[\boldsymbol{S}_x(u_t, v_t) - x_{ic}]^2 + [\boldsymbol{S}_y(u_t, v_t) - y_{ic}]^2 + [\boldsymbol{S}_z(u_t, v_t) - z_{ic}]^2} \end{aligned} \tag{20}$$

5 The NURBS software package for the FANUC 0i-MB/MC/MD CNC system

The FANUC 0i-MB/MC/MD CNC system is widely used in various industries. However, it only provides the two basic interpolations, linear interpolation (G01) and circular interpolation (G02/G03). Consequently, the CNC

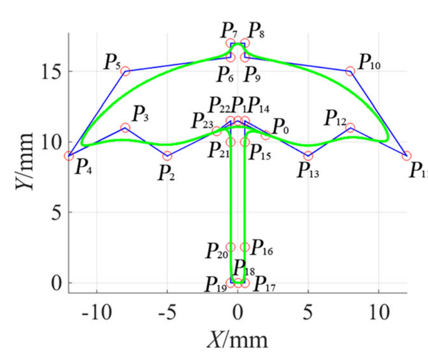


Fig. 5 A planar NURBS curve (like an umbrella)

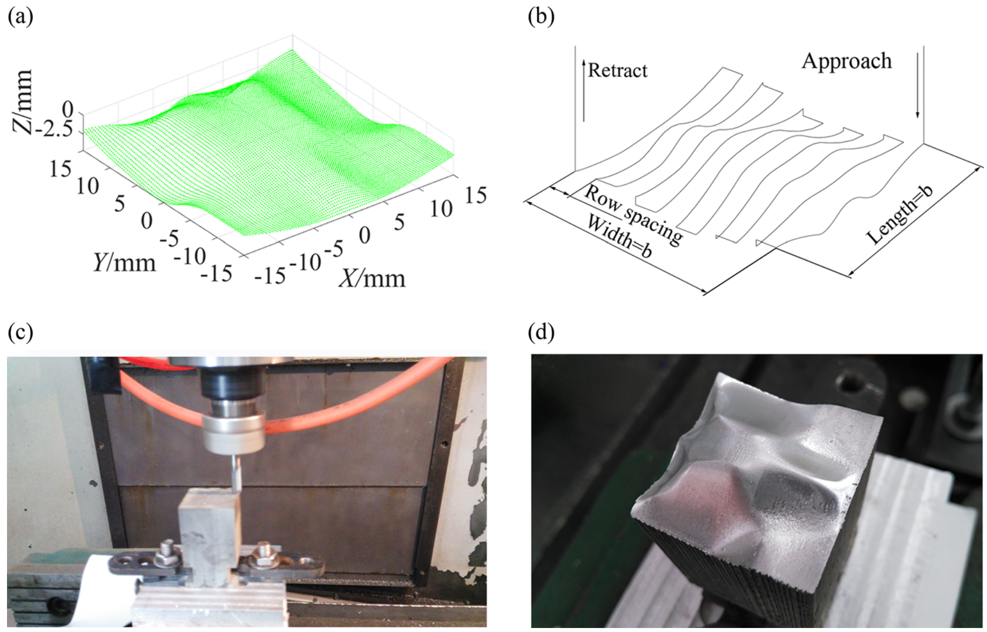


Fig. 6 The design and milling of a NURBS surface. **a** A NURBS surface. **b** The reciprocating milling path of the NURBS surface. **c** The milling process performed on a three-axis CNC milling machine with FANUC 0i-MD CNC system. **d** The final milling sample

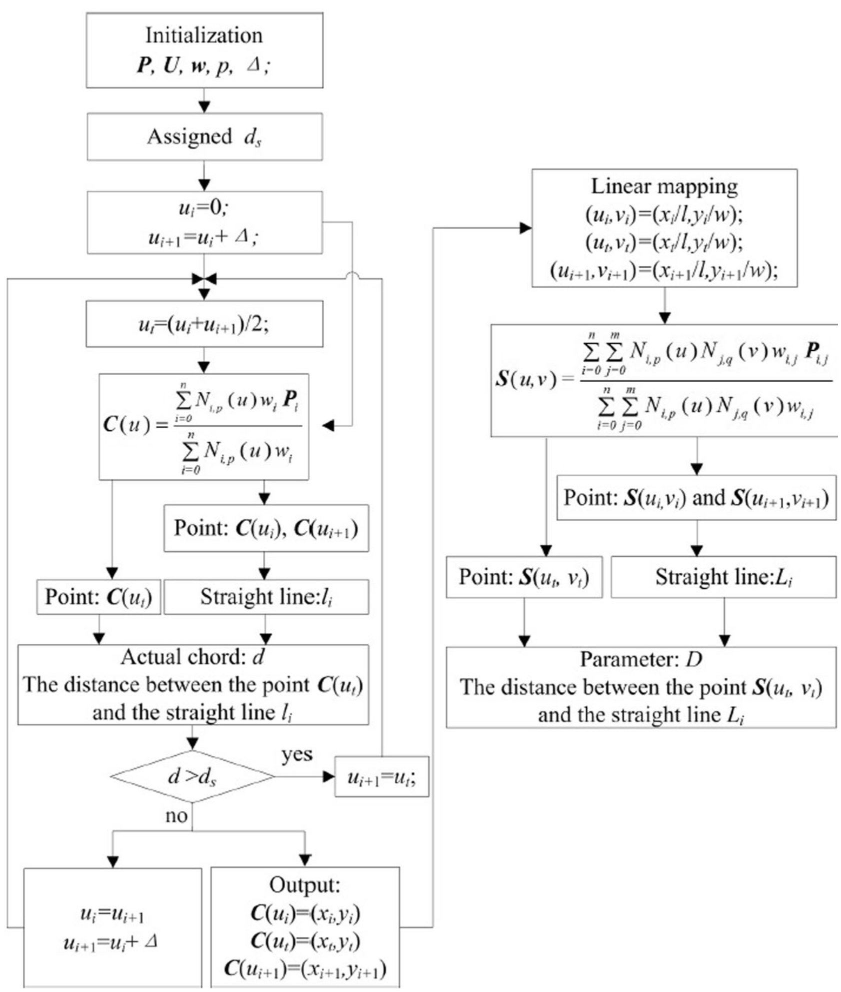
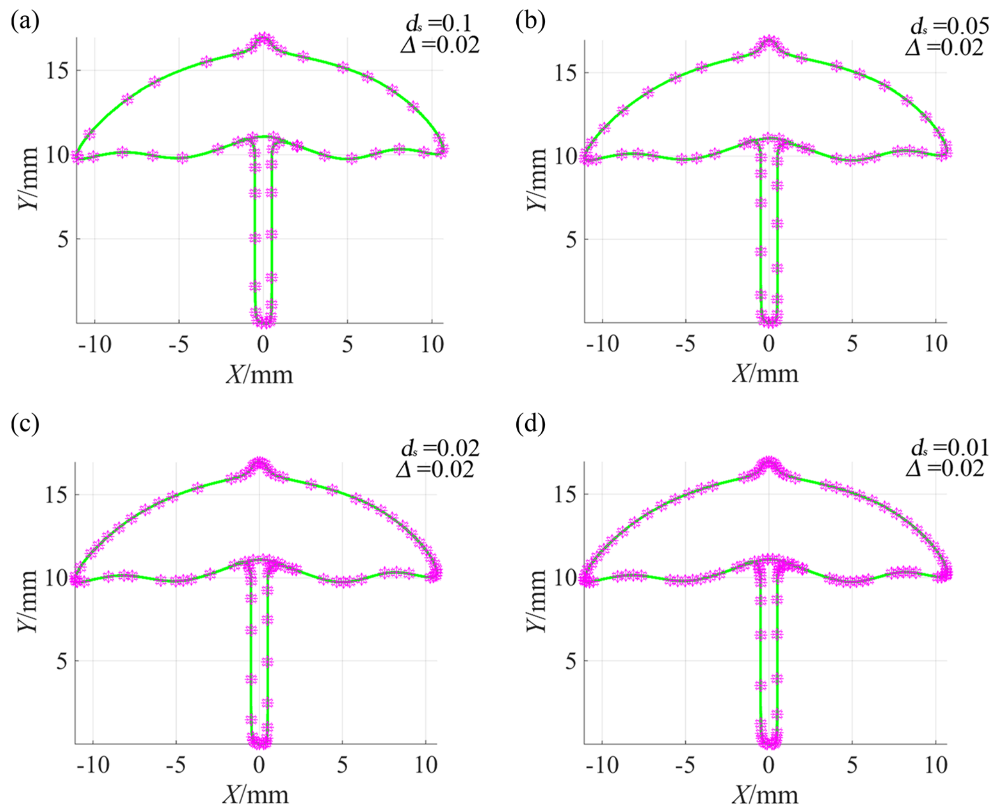


Fig. 7 The flow chart of the bisection method of NURBS-MPCs

Fig. 8 The bisection method results ($\Delta = 0.02$) calculated via NURBS-T. **a** $d_s = 0.1$. **b** $d_s = 0.05$. **c** $d_s = 0.02$. **d** $d_s = 0.01$

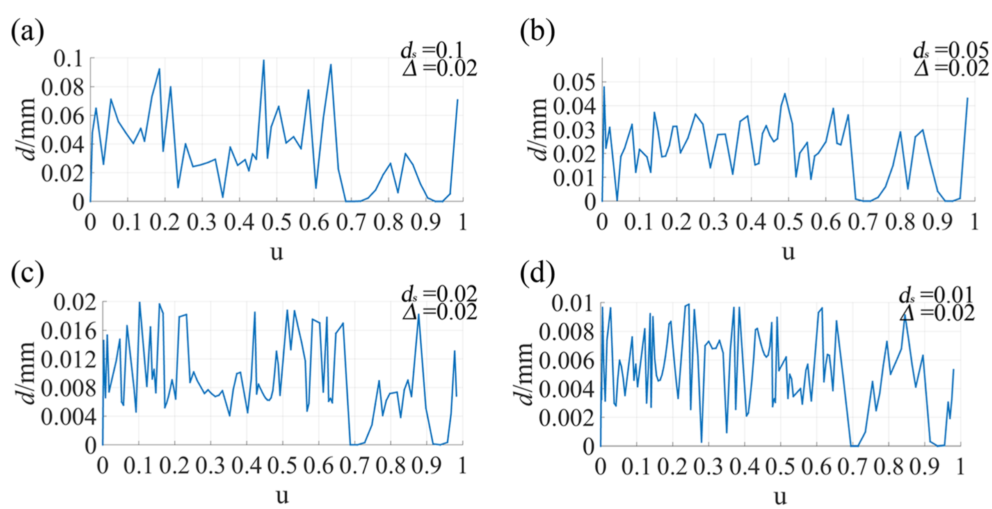


system cannot carry out the milling process of NURBS curves and surfaces for the reason that it does not have the NURBS interpolation function. Fortunately, FANUC 0i-MB/MC/MD provides researchers with the parametric programming function. This function endows great convenience for researchers. Many scholars solve various problems using the parametric programming [19–21].

In this work, based on NURBS theory and the parametric programming of CNC system, an online NURBS software package (NURBS-SP) for FANUC 0i-MB/MC/MD and an offline NURBS toolbox (NURBS-T) based on Matlab software are developed. Figure 3 shows the architecture system of the online NURBS-SP. The corresponding relationship of programs and their variables is illustrated in Fig. 4.

In this work, the research is performed by using NURBS-SP and NURBS-T interactively. The design and simulation are carried out using NURBS-T. Then, the proper values of parameters are assigned to corresponding variables online. Eventually, the milling process is performed by a three-axis CNC milling machine. NURBS-SP endows the low-middle level three-axis CNC system (FANUC 0i-MB/MC/MD) with a powerful NURBS interpolation function.

Fig. 9 The actual chord of d ($\Delta = 0.02$). **a** $d_s = 0.1$. **b** $d_s = 0.05$. **c** $d_s = 0.02$. **d** $d_s = 0.01$



6 The machining of NURBS-MPCs

6.1 The design of a NURBS-MPC

An example of a planar NURBS curve (like an umbrella) is designed as shown in Fig. 5. The control points, weight, and knot vector of the NURBS curve are determined as **Appendix** A.

6.2 The design of a NURBS surface

An example of a NURBS surface is designed as shown in Fig. 6. The degrees (p × q) of this surface is equal to 2 × 2.

The control point, weight, and knot vector of the NURBS surface are assigned as **Appendix** B.

The reciprocating milling program of the NURBS surface is generated by NURBS-T. The milling is performed based on a three-axis CNC milling machine equipped with a FANUC 0i-MD CNC system.

6.3 The bisection method of NURBS-MPCs

In order to accomplish the milling process of a NURBS-MPC, the bisection method is employed. After the initialization as shown in Fig. 7, when the expected chord of d_s is assigned, the algorithm can finish the interpolation procedure for NURBS-MPCs.

With an example of a NURBS curve as shown in Fig. 5, the planar NURBS curve includes the large curvature such as the umbrella surface edges and the top of the umbrella and also includes the small curvature of the umbrella handle. When d_s is defined, the interpolation results and the actual chord of d are shown in Figs. 8 and 9, respectively, under the condition of Δ = 0.02 and d_s = 0.1, 0.05, 0.02, and 0.01. Figure 9 shows that the actual chord d on the NURBS curve of $\boldsymbol{C}(u)$ is less than or equal to d_s.

The parameter D determines the approximation degree of tiny lines to the theoretical NURBS-MPC (as shown in Fig. 2) and the milling precision. The maximum of D ($D_{\max}$) can reflect the interpolation precision and milling precision of the theoretical NURBS-MPC. Figure 10a illustrates the relationship between d_s and $D_{\max}$ with various values of fixed step (Δ). The larger the fixed step (Δ) is, the larger the parameter of $D_{\max}$ is. When Δ is assigned to be a larger value (e.g., 0.1 or 0.05), the $D_{\max}$ curve fluctuates seriously with the increase of d_s. If the expected precision is less than D_R = 0.100 mm, smaller values of Δ should be taken. As can be observed in Fig. 10b, under the condition of Δ = 0.02, $D_{\max} \leq D_R$ = 0.100 mm can be obtained only when d_s is less than 0.02 mm. However, under the condition of Δ = 0.01, $D_{\max} \leq$ 0.100 mm can be gotten in the ranges of $d_s \leq$ 0.02 mm and 0.070 mm $\leq d_s \leq$ 0.100 mm.

Smaller values of Δ and d_s can lead to the increase of the number of mapping projection points (NMPPs) on $\boldsymbol{S}(u,v)$ (as shown in Fig. 1b, Fig. 2, and Fig. 10c). It indicates that the length of the final CNC program becomes longer.

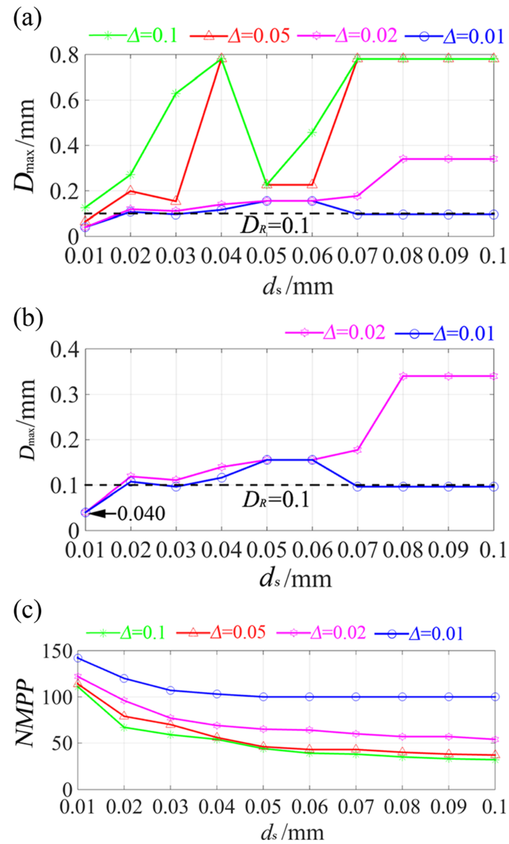


Fig. 10 **a**, **b** The relationship between d_s and $D_{\max}$ with different Δ. **c** The relationship between NMPPs and d_s with various Δ

Obviously, too small value of Δ and d_s can result in the sluggishness of calculating of the NURBS-SP executed by FANUC-0i MB/MC/MD online. In order to strike a balance between the milling precision and the milling efficiency, the ranges of $d_s \leq$ 0.02 (for Δ = 0.01 and 0.02) and $d_s \geq$ 0.07 (for Δ = 0.01) are preferred under the condition of $D_{\max} \leq$ 0.100 mm. Especially, in the two ranges, d_s should be taken larger value.

6.4 The milling process of NURBS-MPCs

The NURBS-MPC of the NURBS curve designed as shown in Fig. 5 is machined based on a three-axis CNC milling machine equipped with a FANUC 0i-MD. Figure 11 shows the milling tool path and the milling results (d_s = 0.08, Δ = 0.01) of the NURBS-MPC using the bisection method based on NURBS-SP executed by FANUC 0i-MD. The milling result proves the effectiveness of the bisection method.

7 Conclusion/recommendation

The interpolation and machining of the projection curve of a NURBS curve is a challenging task. A NURBS mapping

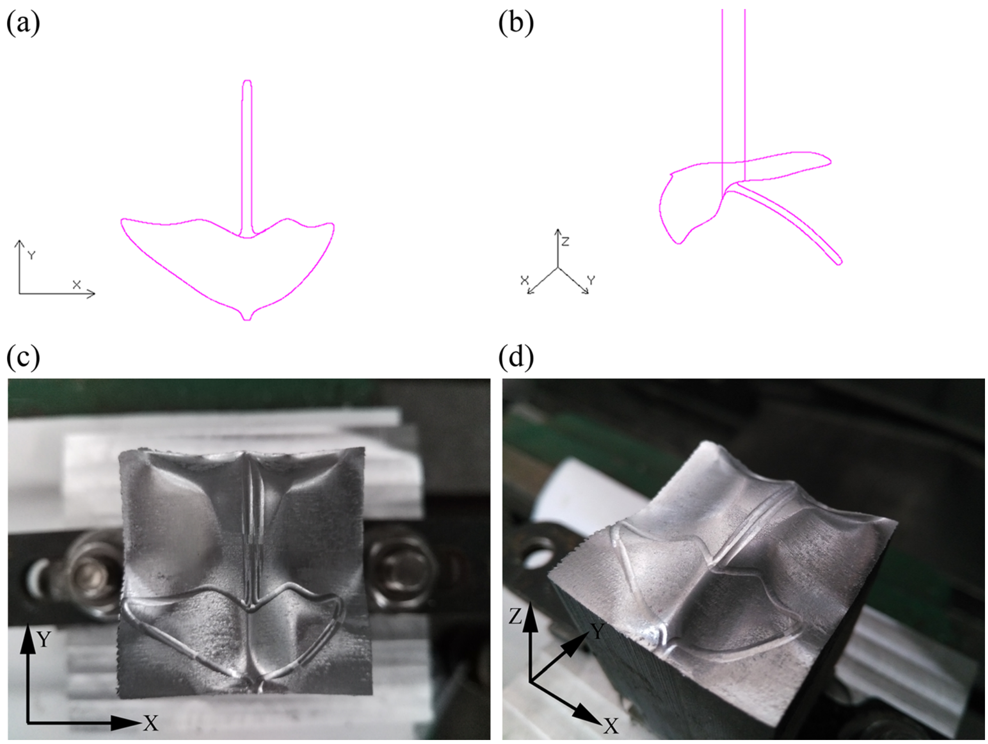


Fig. 11 The milling tool path and sample of the NURBS-MPC using the bisection method based on NURBS-SP executed by FANUC 0i-MD ($d_s = 0.08$, $\Delta = 0.01$). **a** Tool path (XY-view). **b** Tool path (3D–view). **c** The sample (XY-view). **d** The sample (3D–view)

projection curve (NURBS-MPC) is proposed in this work. A NURBS-MPC is defined as a curve on a NURBS surface of a NURBS curve.

In order to accomplish the milling process of NURBS-MPCs, the bisection method is used in this work. Through assigning a proper value of the chord of a NURBS curve, the interpolation and milling of its NURBS-MPC on a NURBS surface can be realized. The bisection method can accomplish the interpolation of a NURBS-MPC. It also can control the interpolation precision of the NURBS-MPC easily.

In order to achieve relative calculation, simulation, and CNC program generation, a NURBS toolbox (NURBS-T) based on Matlab and a NURBS software package (NURBS-SP) for FANUC 0i-MB/MC/MD are developed using the NURBS theory, the bisection method, and the parametric programming method.

With an example of an umbrella (designed using NURBS theory), the simulation and milling of its NURBS-MPC show that the bisection method is feasible and effective. The online NURBS-SP endows the NURBS interpolation function for low-middle level three-axis CNC milling machines and extends the interpolation functions and machining capability of such CNC machines. This feature makes them to have broad application prospects in CNC machining industry.

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Appendix A

The design of a planar NURBS curve (like an umbrella).

- The control points are:

$$\boldsymbol{P} = \{[2\ 10.5\ 0], [0\ 11.5\ 0], [-5\ 9\ 0], [-8\ 11\ 0], [-12\ 9\ 0], [-8\ 15\ 0], [-0.5\ 16\ 0], [-0.5\ 17\ 0], [0.5\ 17\ 0], [0.5\ 16\ 0], [8\ 15\ 0], [12\ 9\ 0], [8\ 11\ 0], [5\ 9\ 0], [0.5\ 11.5\ 0], [0.5\ 10\ 0], [0.5\ 2.5\ 0], [0.5\ 0\ 0], [0\ 0\ 0], [-0.5\ 0\ 0], [-0.5\ 2.5\ 0], [-0.5\ 10\ 0], [-0.5\ 11.5\ 0], [-1.5\ 10.75\ 0]\}$$

- The weight is:

$$\boldsymbol{w} = [1\ 4\ 4\ 3\ 5\ 3\ 5\ 4\ 4\ 5\ 3\ 3\ 3\ 3\ 3\ 5\ 3\ 3\ 1\ 1\ 1\ 3\ 1\ 1]$$

- The knot vector is:

$$\boldsymbol{U} = \left[0\ 0\ 0\ 0\ 1/21\ 2/21\ 3/21\ 4/21\ 5/21\ 6/21\ 7/21\ 8/21\ 9/21\ 10/21\ 11/21\ 12/21\ 13/21\ 14/21\ 15/21\ 16/21\ 17/21\ 18/21\ 19/21\ 20/21\ 1\ 1\ 1\ 1\right]$$

Appendix B

The control points are:

$$P = \{[-15\ 15 -2],\ [-12\ 15 -2],\ [-5\ 15 -5],\ [0\ 15 -2],\ [5\ 15\ -5], [12\ 15 -2], [15\ 15 -2]; [-15\ 12 -3], [-12\ 12 -3], [-5\ 12\ 0],\ [0\ 12 -2],\ [5\ 12\ 0],\ [12\ 12 -3],\ [15\ 12 -3];\ [-15\ 6\ -4], [-12\ 6 -4], [-5\ 6\ 0], [0\ 6\ 0], [5\ 6\ 0], [12\ 6 -4], [15\ 6\ -4]; [-15\ 0 -5], [-12\ 0 -5], [-5\ 0\ 0], [0\ 0\ 0], [5\ 0\ 0], [12\ 0\ -5],\ [15\ 0 -5];\ [-15 -6 -2],\ [-12 -6 -3],\ [-5 -6 -3],\ [0\ -6\ 0],\ [5 -6 -3],\ [12 -6 -3],\ [15 -6 -2];\ [-15 -12 -2],\ [-12\ -12\ -4], [-5 -12 -4], [0 -12 -1], [5 -12 -4], [12 -12 -4], [15\ -12\ -2];\ [-15 -15 -2],\ [-12 -15 -3],\ [-5 -15 -4],\ [0\ -15\ -4],\ [5 -15 -4],\ [12 -15 -3],\ [15 -15 -2]\}$$

- The weight is:

$$w = [1\ 1\ 2\ 2\ 2\ 1\ 1; 1\ 1\ 2\ 2\ 2\ 1\ 1; 1\ 1\ 2\ 2\ 2\ 1\ 1; 1\ 1\ 2\ 2\ 2\ 1\ 1; 1\ 1\ 2\ 2\ 2\ 1\ 1; 1\ 1\ 2\ 2\ 2\ 1\ 1; 1\ 1\ 2\ 2\ 2\ 1\ 1]$$

- The knot vectors are:

$$U = \left[0\ 0\ 0\ 1/5\ 2/5\ 3/5\ 4/5\ 1\ 1\ 1\right]$$
$$V = \left[0\ 0\ 0\ 1/6\ 1/3\ 1/2\ 2/3\ 1\ 1\ 1\right]$$

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