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Some matrix identities on colored Motzkin paths

Sheng-Liang Yang^{a,*}, Yan-Ni Dong^a, Tian-Xiao He^b

^a Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, 730050, Gansu, PR China
 ^b Department of Mathematics, Illinois Wesleyan University, Bloomington, IL, 61702–2900, USA

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ABSTRACT

Merlini and Sprugnoli (2017) give both an algebraic and a combinatorial proof for an identity proposed by Louis Shapiro by using Riordan arrays and a particular model of lattice paths. In this paper, we revisit the identity and emphasize the use of colored partial Motzkin paths as appropriate tool. By using colored Motzkin paths with weight defined according to the height of its last point, we can generalize the identity in several ways. These identities allow us to move from Fibonacci polynomials, Lucas polynomials, and Chebyshev polynomials, to the polynomials of the form $(z + b)^n$.

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1. Introduction

In [25], Shapiro introduced a triangle $(B_{n,k})_{n,k\geq 0}$, where $B_{n,k} = \frac{k+1}{n+1} \binom{2n+2}{n-k}$. Then the following identity related with this matrix was obtained in Shapiro et al. [27]

$$\sum_{k=0}^{n} B_{n,k}(k+1) = 4^{n}.$$
(1.1)

The entries of this matrix have the following combinatorial interpretation (see, for example, [25,36]). Consider a pair of paths that start at the origin, consist of n + 1 unit steps either east E = (1, 0) or north N = (0, 1), finishing at the points (a, b) and (c, d). It is assumed that the two paths do not meet after (0, 0). Then $B_{n,k}$ counts the number of such pairs of paths for which c - a = k + 1, i.e., which end a (horizontal) distance k + 1. Call these partial path pairs. By using this combinatorial interpretation, Woan et al. [36] have given an elegant proof of the above identity. Some other combinatorial interpretations of the identity were given in Callan [5], Cameron and Nkwanta [6], and Chen et al. [7]. Very recently, Merlini and Sprugnoli [18] give both an algebraic and a combinatorial proof for this identity by using Riordan arrays [26] and a particular model of lattice paths, and they also find several generalizations of this identity and obtain a general transformation from an arithmetic into a geometric progression.

In this paper, we will give a combinatorial interpretation and many generalizations of identity (1.1) by using colored partial Motzkin paths and Riordan arrays. A Motzkin path of length n is a lattice path from (0, 0) to (n, 0) consisting of up steps U = (1, 1), horizontal steps H = (1, 0) and down steps D = (1, -1) that never goes below the x-axis. A (u, h, d)-colored Motzkin path is a Motzkin path such that the up steps, horizontal steps and down steps are labeled by u colors, h colors and d colors, respectively. In the literatures, the (1, h, 1)-colored Motzkin paths are called the h-colored Motzkin paths, while

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^{*} Corresponding author. E-mail address: slyang@lut.cn (S.-L. Yang).

the (1, h, d)-colored Motzkin paths are called the (h, d)-colored Motzkin paths [7, 12, 24, 35]. It is well known that the set of Motzkin paths of length *n* is enumerated by the Motzkin numbers M_n with generating function $M(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$, and the set of 2-colored Motzkin paths of length *n* is counted by the Catalan numbers C_{n+1} . A (u, h, d)-colored partial Motzkin path, also called a (u, h, d)-colored Motzkin path ending an (n, k), is defined as an initial segment of a (u, h, d)-colored Motzkin path with terminal point (n, k). Let $\mathcal{M}_{n,k}$ denote the set of all 2-colored Motzkin paths ending at (n, k), where $\mathcal{M}_{0,0} = \{\varepsilon\}$ and ε is the empty path. Cameron and Nkwanta [6] showed that the (n, k)th entry of Shapiro's matrix in identity (1.1) is $|\mathcal{M}_{n,k}|$, and they presented a combinatorial proof of this identity. Chen et al. [7] find many extensions of identity (1.1)by using colored Motzkin paths.

In the next section, by using the method of Merlini and Sprugnoli [19] and the 2-colored Motzkin paths, we obtain a combinatorial proof of identity (1.1). In addition, we establish a bijection ϕ between the set $\mathcal{B}_{n,k}$ of partial path pairs of length n + 1 and distance k + 1 and the set $\mathcal{M}_{n,k}$ of 2-colored Motzkin paths ending at (n, k). Then by considering (1, b, c)-colored Motzkin paths, we are able to get identities involving the Fibonacci polynomials and the sequence $(1, z + b, (z + b)^2, ...)$. In Section 3, we consider three kinds of colored Motzkin paths with privileged steps on the x-axis, and we obtain further identities involving the Lucas polynomials and Chebyshev polynomials.

At the end of this section, we briefly recall the notion of Riordan arrays [8,14,26,29]. An infinite lower triangular matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column *k* has generating function $d(t)h(t)^k$, where $d(t) = \sum_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum_{n=1}^{\infty} h_n t^n$ are formal power series with $d_0 \neq 0$ and $h_1 \neq 0$. The Riordan array corresponding to the pair d(t) and h(t) is denoted by (d(t), h(t)), and its generic entry is $g_{n,k} = [t^n]d(t)h(t)^k$, where $[t^n]$ denotes the coefficient operator.

The set of all proper Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity (1, t). The product of two Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))),$$
(1.2)

and the inverse of (d(t), h(t)) is the Riordan array

$$(d(t), h(t))^{-1} = (1/d(\bar{h}(t)), \bar{h}(t)),$$
(1.3)

where $\bar{h}(t)$ is compositional inverse of h(t), i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = t$. If $(b_n)_{b \in \mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array (d(t), h(t)) = b. $(g_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} g_{n,k} b_k = [t^n] d(t) b(h(t)).$$
(1.4)

This is called the fundamental theorem of Riordan arrays and it can be rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)).$$
(1.5)

A Riordan array $G = (d(t), h(t)) = (g_{n,k})_{n,k \in \mathbb{N}}$ can be characterized [14,16,23,29] by two sequences, the A-sequence, $A = (a_n)_{n \in \mathbb{N}}$ and, the *Z*-sequence, $Z = (z_n)_{n \in \mathbb{N}}$ such that

$$g_{n+1,0} = z_0 g_{n,0} + z_1 g_{n,1} + z_2 g_{n,2} + \cdots + z_n g_{n,n},$$

$$g_{n+1,k+1} = a_0 g_{n,k} + a_1 g_{n,k+1} + a_2 g_{n,k+2} + \dots + a_{n-k} g_{n,n}$$

for all n, k > 0. If A(t) and Z(t) are the generating functions for the corresponding A- and Z-sequences, respectively, then it follows that

$$d(t) = \frac{1}{1 - tZ(h(t))}, \text{ and } h(t) = tA(h(t)).$$
 (1.6)

Furthermore, if the inverse of (d(t), h(t)) is $(d(t), h(t))^{-1} = (g(t), f(t))$, then we have

$$f(t) = \frac{t}{A(t)}$$
, and $g(t) = 1 - \frac{tZ(t)}{A(t)}$. (1.7)

For example, the Shapiro's array in identity (1.1) corresponds to the Riordan array $B = (C(t)^2, tC(t)^2)$, where C(t) = $\frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers. By the properties of Riordan arrays, the identity (1.1) can be rewritten as

$$\left(C(t)^2, tC(t)^2\right) \frac{1}{(1-t)^2} = \frac{1}{1-4t}.$$
(1.8)

By using Riordan arrays and a particular model of lattice paths, Merlini and Sprugnoli [19] have given both an algebraic and a combinatorial proof of the following identity (see also [11])

$$\left(C(t), tC(t)^2\right) \frac{1+t}{(1-t)^2} = \frac{1}{1-4t}.$$
(1.9)

We will present a combinatorial interpretation of the matrix identity (1.8) by using the 2-colored Motzkin paths.



Fig. 1. The inductive step.

2. Colored Motzkin paths

Throughout this paper, we will consider the classes of (u, h, d)-colored Motzkin paths described by the following points:

- 1. the path is composed of up steps U = (1, 1), horizontal steps H = (1, 0), and down steps D = (1, -1);
- 2. up steps occur in *u* colors, horizontal steps in *h* colors, and down steps in *d* colors;
- 3. paths start at the origin and remain in the first octant, that is, for every point (x, y) in the path, we have 0 < y < x;
- 4. each path is endowed with weight w_k , where k is the vertical distance of its last point from the x-axis, i.e., a path ending at point (n, k) has weight w_k , where $\{w_k\}_{k>0}$ is the weight sequence.

2.1. (1, 2, 1)-colored Motzkin paths

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Theorem 2.1. Let $R_{n,k}$ be the number of (1, 2, 1)-colored Motzkin paths ending at (n, k) and having weight $w_k = k + 1$. Then

$$\sum_{k=0} R_{n,k}(k+1) = 4^n, \quad \forall n \in \mathbb{N},$$
(2.1)

that is, the total weight of the paths of length n is 4^n .

Proof. We prove the theorem by induction on *n*. For n = 0 the only existing path reduces to the origin; therefore its weight is $1 = 4^0$.

Assume that the equation is true for some $n \ge 1$, and add a new step in all possible ways to all the paths of length n. These paths terminate at a point (n, k). In the case k > 0, as we see in Fig. 1, for every path (O, P) of weight p = k + 1 we can:

- add an up step, producing a path of length n + 1 with weight p + 1;
- add a horizontal step, producing a path of length n + 1 and weight p; this can be done with a horizontal step in 2 colors;
- add a down step, producing a path of length n + 1 and weight p 1.

The total balance is (p + 1) + 2 * p + (p - 1) = 4p. For paths of length *n* and ending on the *x*-axis, we only can add an up or a horizontal step with 2 colors. In the former case the weight grows to 2. In the latter case, adding a horizontal step the weight does not change remaining at 1, but this can be done in 2 colors, so the total balance is 2 + 2, which is 4p, being p = 1 the initial weight. In every case, the total weight grows by 4 times, which proves the induction step. \Box

Theorem 2.2. For $n \ge k \ge 0$, there is a bijection ϕ between the set $\mathcal{B}_{n,k}$ of partial path pairs of length n + 1 and distance k + 1 and the set $\mathcal{M}_{n,k}$ of (1, 2, 1)-colored Motzkin paths ending at (n, k).

Proof. For a partial path pair (P_1, P_2) in $\mathcal{B}_{n,k}$, assume $P_1 = X_0 X_1 X_2 \cdots X_n$ and $P_2 = Y_0 Y_1 Y_2 \cdots Y_n$ with X_i and Y_i are N = (0, 1) or E = (1, 0) for each *i* except that $X_0 = N$ and $Y_0 = E$. We may write this partial path pair as a sequence of step pairs

$$P_1, P_2) = (X_0, Y_0)(X_1, Y_1)(X_2, Y_2) \cdots (X_n, Y_n).$$

Define

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$$\phi(P_1, P_2) = \begin{cases} \varepsilon, \text{ if } n = 0, \\ \phi(X_1, Y_1)\phi(X_2, Y_2) \cdots \phi(X_n, Y_n), \text{ if } n > 0, \end{cases}$$

where ε is the empty path and for each i > 0

$$\phi(X_i, Y_i) = \begin{cases} U, & \text{if } (X_i, Y_i) = (N, E), \\ D, & \text{if } (X_i, Y_i) = (E, N), \\ H_1, & \text{if } (X_i, Y_i) = (E, E), \\ H_2, & \text{if } (X_i, Y_i) = (N, N) \end{cases}$$

Then $\phi(\mathcal{B}_{0,0}) = \{\varepsilon\}$, and $\phi(\mathcal{B}_{n,k})$ is the set of all (1, 2, 1)-Motzkin paths ending at (n, k) for $n \ge 1$. Obviously, the map ϕ is a bijection. \Box

2.2. (1, b, c)-colored Motzkin paths

The concepts of Fibonacci polynomials and Lucas polynomials are very old and well known [9,32,33]. The Fibonacci polynomials $f_n(z, c) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} {\binom{n-k-1}{k}} z^{n-2k-1} c^k$ satisfy the recursion $f_n(z, c) = zf_{n-1}(z, c) + cf_{n-2}(z, c)$ for $n \ge 2$ with initial values $f_0(z, c) = 0$ and $f_1(z, c) = 1$. We will consider the special Fibonacci polynomials $F_n(z, c) = f_{n+1}(z, -c)$, their generating function is $\sum_{n=0}^{\infty} F_n(z, c)t^n = \frac{1}{1-zt+ct^2}$.

Theorem 2.3. Let $R_{n,k}$ be the number of (1, b, c)-colored Motzkin paths ending at (n, k) and having weight $F_k(z, c)$. Then

$$\sum_{k=0}^{\infty} R_{n,k} F_k(z,c) = (z+b)^n, \quad \forall n \in \mathbb{N},$$
(2.2)

that is, the total weight of the paths of length n is $(z + b)^n$.

Proof. Let *p* be the weight of a path ending at a node (n, k) with k > 0; now $p = F_k(z, c)$, every up step changes the weight to $F_{k+1}(z, c)$; every horizontal step leaves the weight unchanged; finally, every down step changes the weight to $F_{k-1}(z, c)$. So, the total balance is $F_{k+1}(z, c) + bF_k(z, c) + cF_{k-1}(z, c) = (zF_k(z, c) - cF_{k-1}(z, c)) + bF_k(z, c) + cF_{k-1}(z, c) = (z + b)F_k(z, c)$, i.e., the total weight (relative to internal nodes) has increased (z + b) times.

For what concerns the nodes on the *x*-axis, their weight is initially $F_0(z, c) = 1$, then: the *b* horizontal steps do not change this weight, so they contribute with *b* units to the total balance; one up step contributes with *z* to the total balance since every up step changes the weight to $F_1(z, c) = z$. Therefore, the total balance is z + b, it also increases (z + b) times. \Box

The matrix form of (2.2) is

 $(R_{i,i})_{i,i>0} * (F_k(z, c))_k = ((z+b)^n)_n.$

Writing the first few rows of $(R_{i,i})_{i,i>0}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ b^2 + c & 2b & 1 & 0 & 0 \\ b^3 + 3bc & 3b^2 + 2c & 3b & 1 & 0 \\ b^4 + 6b^2c + 2c^2 & 4b^3 + 8bc & 6b^2 + 3c & 4b & 1 \end{pmatrix} \begin{pmatrix} 1 \\ F_1(z,c) \\ F_2(z,c) \\ F_3(z,c) \\ F_4(z,c) \end{pmatrix} = \begin{pmatrix} 1 \\ z+b \\ (z+b)^2 \\ (z+b)^3 \\ (z+b)^4 \end{pmatrix}.$$

By specifying the values of b, c and z in the above matrix identity, we obtain many interesting relations.

Example 2.1. (i) For b = 2, c = 1, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 & 0 \\ 42 & 48 & 27 & 8 & 1 & 0 \\ 132 & 165 & 110 & 44 & 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ F_1(z,1) \\ F_2(z,1) \\ F_3(z,1) \\ F_4(z,1) \\ F_5(z,1) \end{pmatrix} = \begin{pmatrix} 1 \\ z+2 \\ (z+2)^2 \\ (z+2)^2 \\ (z+2)^3 \\ (z+2)^4 \\ (z+2)^5 \end{pmatrix},$$

this is a direct generalization of (1.1).

(ii) For b = 0, c = 1, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 4 & 0 & 1 & 0 \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ F_1(z,1) \\ F_2(z,1) \\ F_3(z,1) \\ F_4(z,1) \\ F_5(z,1) \\ F_6(z,1) \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \end{pmatrix},$$

where the matrix is the aerated Catalan triangle [3,34], and $F_n(z, 1)$ are the special Fibonacci polynomials [9]. (iii) For b = 3, c = 2, z = 3, we obtain the identity (1.3) in Chen [7]:

(1	0	0	0	0	0/	(1)		(1)	1
3	1	0	0	0	0	3		6	
11	6	1	0	0	0	7		6 ²	
45	31	9	1	0	0	15	=	6 ³	ľ
197	156	60	12	1	0	31		6 ⁴	
(903	785	360	98	15	1/	(63)		\6 ⁵ /	/

where the first column is the sequence of little Schröder numbers.

(iv) For c = 1, z = 3, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ b^2 + 1 & 2b & 1 & 0 & 0 \\ b^3 + 3b & 3b^2 + 2 & 3b & 1 & 0 \\ b^4 + 6b^2 + 2 & 4b^3 + 8b & 6b^2 + 3 & 4b & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 21 \\ 55 \end{pmatrix} = \begin{pmatrix} 1 \\ b + 3 \\ (b + 3)^2 \\ (b + 3)^3 \\ (b + 3)^4 \end{pmatrix}.$$

where the sequence (1, 3, 8, 21, 55, ...) consists of Fibonacci numbers of even index. A special form of this matrix identity for b = 2 is obtained in [1].

(v) For c = t, z = t + 1 and b = k - t - 1, we obtain the identity found in [7]:

$$(R_{i,j})_{i,j\geq 0} \begin{pmatrix} 1\\ 1+t\\ 1+t+t^2\\ 1+t+t^2+t^3\\ \vdots \end{pmatrix} = \begin{pmatrix} 1\\ k\\ k^2\\ k^3\\ \vdots\\ \vdots \end{pmatrix},$$

where $R_{i,j}$ is the number of (1, k - t - 1, t)-colored Motzkin paths ending at (n, k) and having weight $F_k(t + 1, t) = 1 + t + \cdots + t^k$.

3. Colored Motzkin paths with privileged steps

In this section, we will consider the classes of (u, h, d)-colored Motzkin paths satisfying the following additional conditions:

- 5. each horizontal step on the *x*-axis in h_0 colors;
- 6. each down step ended on the *x*-axis in d_0 colors;

These paths are said to have privileged steps to the *x*-axis [17]. In the sequel, these paths will be called $(u, h + h_0, d + d_0)$ -colored Motzkin paths, and we will prove our results by an algebraic approach.

3.1. (1, b, c + 2c) -colored Motzkin paths

The Lucas polynomials $l_n(z, c) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} {n-k \choose k} z^{n-2k} c^k$ satisfy the recursion $l_n(z, c) = z l_{n-1}(z, c) + c l_{n-2}(z, c)$ for $n \ge 2$ with initial values $l_0(z, c) = 2$ and $l_1(z, c) = z$. We will consider the normalized Lucas polynomials $L_n(z, c) = l_n(z, -c)$ for $n \ge 1$ and $L_0(z, c) = 1$, their generating function is $\sum_{n=0}^{\infty} L_n(z, c)t^n = \frac{1-ct^2}{1-zt+ct^2}$.

Theorem 3.1. Let $R_{n,k}$ be the number of (1, b, c + 2c)-colored Motzkin paths ending at (n, k) and having weight $L_k(z, c)$ for $k \ge 0$, then

$$\sum_{k=1}^{n} R_{n,k} L_k(z,c) = (z+b)^n, \quad \forall n \in \mathbb{N},$$
(3.1)

that is, the total weight of the paths of length n is $(z + b)^n$.

Proof. Analogously the numbers $R_{n,k}$ satisfy the following recurrence relations,

 $R_{n+1,0} = bR_{n,0} + 2cR_{n,1},$ $R_{n+1 k+1} = R_{n k} + bR_{n k+1} + cR_{n k+2}.$

This implies that the infinite triangle $R = (R_{n,k})_{n,k \in \mathbb{N}}$ is a Riordan array and the generating functions for the A- and Z-sequences are $A(t) = 1 + bt + ct^2$ and Z(t) = b + 2ct, respectively. Applying (1.3) and (1.7), we get

$$R^{-1} = \left(\frac{1 - ct^2}{1 + bt + ct^2}, \frac{t}{1 + bt + ct^2}\right),$$
$$R = \left(\frac{1}{\sqrt{(1 - bt)^2 - 4ct^2}}, \frac{1 - bt - \sqrt{(1 - bt)^2 - 4ct^2}}{2ct}\right)$$

Thus, the identity (3.1) can now be written as $R * (L_k)_k = ((z+b)^n)_n$, which is equivalent to $R^{-1} * ((z+b)^k)_k = (L_n)_n$, where $L_0 = 1, L_1 = z, L_2 = z^2 - 2c$, and $L_n = zL_{n-1} - cL_{n-2}$ for $n \ge 3$. This is easier to prove: $\left(\frac{1-ct^2}{1+bt+ct^2}, \frac{t}{1+bt+ct^2}\right) * \frac{1}{1-(z+b)t} = \frac{1-ct^2}{1+bt+ct^2} \cdot \frac{1+bt+ct^2}{1+bt+ct^2} - \frac{1-ct^2}{1-zt+ct^2}$, and $\frac{1-ct^2}{1-zt+ct^2} = \sum_{n=0}^{\infty} L_n(z, c)t^n$.

Example 3.1. For b = 2, c = 1, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 & 0 \\ 20 & 15 & 6 & 1 & 0 & 0 \\ 70 & 56 & 28 & 8 & 1 & 0 \\ 252 & 210 & 120 & 45 & 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ L_1(z,1) \\ L_2(z,1) \\ L_4(z,1) \\ L_5(z,1) \end{pmatrix} = \begin{pmatrix} 1 \\ z+2 \\ (z+2)^2 \\ (z+2)^3 \\ (z+2)^4 \\ (z+2)^5 \end{pmatrix}.$$

The above matrix is the central binomial triangle [4,15,20] whose first column is the sequence of central binomial coefficients.

Theorem 3.2. Let $R_{n,k}$ be the number of (1, b, c+2c)-colored Motzkin paths ending at (n, k) and having weight $c^k + 1$ for $k \ge 1$ and 1 for k = 0, then

$$R_{n,0} + \sum_{k=1}^{n} R_{n,k} (c^k + 1) = (b + c + 1)^n, \quad \forall n \in \mathbb{N},$$
(3.2)

that is, the total weight of the paths of length n is $(b + c + 1)^n$.

Proof. Since $\frac{1-ct^2}{1-(c+1)t+ct^2} = 1 + \frac{t}{1-ct} + \frac{ct}{1-ct}$ is the generating function of the sequence 1, c + 1, $c^2 + 1$, Hence, setting z = c + 1 in Theorem 2.3 we obtain the result. \Box

Example 3.2. (i) For b = 0, c = 1, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 6 & 0 & 4 & 0 & 1 & 0 \\ 0 & 10 & 0 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2^2 \\ 2^3 \\ 2^4 \\ 2^5 \end{pmatrix}.$$

The above matrix is the aeration of right half of binomial triangle [9,37] whose first column is the aeration of sequence of central binomial coefficients.

(ii) For b = 1, c = 1, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 7 & 6 & 3 & 1 & 0 & 0 \\ 19 & 16 & 10 & 4 & 1 & 0 \\ 51 & 45 & 30 & 15 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3^2 \\ 3^4 \\ 3^5 \end{pmatrix}.$$

The above matrix is the right half of trinomial triangle [18,21] whose first column is the sequence of central trinomial coefficients.

(iii) For b = 3, c = 2, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 13 & 6 & 1 & 0 & 0 & 0 \\ 63 & 33 & 9 & 1 & 0 & 0 \\ 321 & 180 & 62 & 12 & 1 & 0 \\ 1683 & 985 & 390 & 100 & 15 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \\ 9 \\ 17 \\ 33 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 6^2 \\ 6^3 \\ 6^4 \\ 6^5 \end{pmatrix}$$

The above matrix is included in Peart and Shapiro [21], where the first column is the sequence of central Delannoy numbers [31].

Theorem 3.3. Let $R_{n,k}$ be the number of $(1, b, e^2 + 2e^2)$ -colored Motzkin paths ending at (n, k) and having weight $2e^k$ for $k \ge 1$ and 1 for k = 0, then

$$R_{n,0} + \sum_{k=1}^{n} R_{n,k} (2e^k) = (b+2e)^n, \ \forall n \in \mathbb{N},$$
(3.3)

that is, the total weight of the paths of length n is $(b + 2e)^n$.

Proof. Since $\frac{1-e^2t^2}{1-2et+e^2t^2} = \frac{1+et}{1-et}$ is the generating function of the sequence 1, 2e, 2e², Hence, substituting $c = e^2$ and z = 2e in Theorem 2.3 yields the result. \Box

Example 3.3. (i) For b = 4, e = 2, we obtain the identity:

(1	0	0	0	0	0)	(1)		(1)	
4	1	0	0	0	0	4		8	
24	8	1	0	0	0	8		8 ²	
160	60	12	1	0	0	16	=	8 ³	ŀ
1120	448	112	16	1	0	32		8 ⁴	
8064	3360	960	180	20	1)	\6 4/		\ ₈₅	

The above matrix is A128417 in [28].

(ii) For b = 1, e = 2, we obtain the identity:

(1	0	0	0	0	0)	(1)		$(^{1})$	
1	1	0	0	0	0	4		5	
9	2	1	0	0	0	8		5^{2}	
25	15	3	1	0	0	16	=	5 ³	
145	52	22	4	1	0	32		5 ⁴	
561	285	90	30	5	1/	64/		\ _{5⁵} /	

The first column of the above matrix is A084605 in [28].

3.2. (1, b + (b + e), c)-colored Motzkin paths

The Chebyshev polynomials have many beautiful properties and countless applications [10,13]. The Chebyshev polynomials of the first kind are defined by $T_0(z) = 1$, $T_1(z) = z$, and for $n \ge 2$, $T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)$. Their generating function is The third and fourth kinds of Chebyshev polynomials are defined by generating functions $\sum_{n=0}^{\infty} U_n(z)t^n = \frac{1-zt}{1-2zt+t^2}$. The Chebyshev polynomials of the second kind differ only in the initial conditions. They are defined by $U_0(z) = 1$, $U_1(z) = 2z$, and for $n \ge 2$, $U_n(z) = 2zU_{n-1}(z) - U_{n-2}(z)$. Their generating function is $\sum_{n=0}^{\infty} U_n(z)t^n = \frac{1}{1-2zt+t^2}$. The third and fourth kinds of Chebyshev polynomials are defined by generating functions $\sum_{n=0}^{\infty} V_n(z)t^n = \frac{1-t}{1-2zt+t^2}$, and $\sum_{n=0}^{\infty} W_n(z)t^n = \frac{1+t}{1-2zt+t^2}$, respectively (see [2]). Now, we define the generalized Chebyshev polynomials by generating function

$$\sum_{n=0}^{\infty} g_n(z, c, e)t^n = \frac{1-et}{1-zt+ct^2}.$$

Then they satisfy the recursion $g_n(z, c, e) = zg_{n-1}(z, c, e) - cg_{n-2}(z, c, e)$ with initial conditions $g_0(z, c, e) = 1$ and $g_1(z, c, e) = z - e.$

Theorem 3.4. Let $R_{n,k}$ be the number of (1, b + (b + e), c)-colored Motzkin paths ending at (n, k) and having weight $g_k(z, c, e)$. Then

$$\sum_{k=0}^{n} R_{n,k} g_k(z,c,e) = (z+b)^n, \quad \forall n \in \mathbb{N},$$
(3.4)

that is, the total weight of the paths of length n is $(z + b)^n$.

Proof. The numbers $R_{n,k}$ satisfy the following recurrence relations,

$$R_{n+1,0} = (b+e)R_{n,0} + cR_{n,1},$$

$$R_{n+1,k+1} = R_{n,k} + bR_{n,k+1} + cR_{n,k+2}.$$

Hence the infinite triangle $R = (R_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array and the generating functions for the A- and Z-sequences are $A(t) = 1 + bt + ct^2$ and Z(t) = b + e + ct, respectively. Using (1.3) and (1.7), we obtain

$$R^{-1} = \left(\frac{1 - et}{1 + bt + ct^2}, \frac{t}{1 + bt + ct^2}\right),$$

$$R = \left(\frac{2}{1 - (2e + b)t + \sqrt{(1 - bt)^2 - 4ct^2}}, \frac{1 - bt - \sqrt{(1 - bt)^2 - 4ct^2}}{2ct}\right).$$

Thus, the identity (3.3) can now be written as $R * (g_k(z, c, e))_k = ((b+z)^n)_n$, which is equivalent to $R^{-1} * ((b+z)^k)_k = (g_n(z, c, e))_n$. This is easier to prove: $\left(\frac{1-et}{1+bt+ct^2}, \frac{t}{1+bt+ct^2}\right) * \frac{1}{1-(b+z)t} = \frac{1-et}{1+bt+ct^2} \cdot \frac{1+bt+ct^2}{1+bt+ct^2-(b+z)t} = \frac{1-et}{1-zt+ct^2}$. \Box

Example 3.4. (i) For b = 2, c = 1, e = 1, and z = 2x, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 10 & 5 & 1 & 0 & 0 & 0 \\ 35 & 21 & 7 & 1 & 0 & 0 \\ 126 & 84 & 36 & 9 & 1 & 0 \\ 462 & 330 & 156 & 55 & 11 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ V_1(x) \\ V_2(x) \\ V_3(x) \\ V_4(x) \\ V_5(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 2x+2 \\ (2x+2)^2 \\ (2x+2)^2 \\ (2x+2)^4 \\ (2x+2)^5 \end{pmatrix},$$

where $V_n(x)$ are the modified Chebyshev polynomials of the third kind. The above matrix is mentioned in [4,20]. (ii) For b = 3, c = 2, e = -2, z = 3, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 11 & 17 & 7 & 1 & 0 & 0 \\ 45 & 76 & 40 & 10 & 1 & 0 \\ 197 & 353 & 216 & 72 & 13 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 13 \\ 29 \\ 61 \\ 125 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 6^2 \\ 6^3 \\ 6^4 \\ 6^5 \end{pmatrix},$$

where $g_0 = 1$, $g_1 = 5$, $g_k = 3g_{k-1} - 2g_{k-2}$ for $k \ge 2$. The above matrix is mentioned in [9], and its first column is the sequence of little Schröder numbers [30].

(iii) For b = 3, c = 2, e = -1, z = 3, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 1 & 0 & 0 \\ 22 & 23 & 8 & 1 & 0 \\ 90 & 107 & 49 & 11 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 13 \\ 29 \\ 61 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 7^2 \\ 7^3 \\ 7^4 \end{pmatrix},$$

where $g_0 = 1$, $g_1 = 5$, $g_k = 4g_{k-1} - 2g_{k-2}$ for $k \ge 2$. The above matrix is A133367 in [28], and its first column is the sequence of large Schröder numbers [30]. A triple factorization of this matrix was found in [22].

(iv) For c = 1, e = 1, z = 3, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ b+1 & 1 & 0 & 0 \\ b^2+2b+2 & 2b+1 & 1 & 0 \\ b^3+3b^2+6b+3 & 3b^2+3b+3 & 3b+1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ b+3 \\ (b+3)^2 \\ (b+3)^3 \end{pmatrix}.$$

where $g_0 = 1$, $g_1 = 2$, $g_k = 3g_{k-1} - g_{k-2}$ for $k \ge 2$, are Fibonacci numbers of odd index.

Corollary 3.5. Let $R_{n,k}$ be the number of $(1, b + (b + d), r^2)$ -colored Motzkin paths ending at (n, k) and having weight $(k + 1)r^k - kdr^{k-1}$, then

$$\sum_{k=0}^{n} R_{n,k}((k+1)r^k - kdr^{k-1}) = (b+2r)^n, \quad \forall n \in \mathbb{N},$$
(3.5)

that is, the total weight of the paths of length n is $(b + 2r)^n$.

Proof. By writing

$$\sum_{k=0}^{\infty} g_k(2r, r^2, d)t^k = \frac{1-dt}{1-2rt+r^2t^2} = \frac{1-dt}{(1-rt)^2},$$

we see that $g_k(2r, r^2, d) = (k + 1)r^k - kdr^{k-1}$. Thus, setting $c = r^2$ and z = 2r in Theorem 3.4 gives us the result. \Box

For example, in the case b = 4, d = r = 2, we obtain the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 \\ 40 & 10 & 1 & 0 & 0 & 0 \\ 280 & 84 & 14 & 1 & 0 & 0 \\ 2016 & 672 & 144 & 18 & 1 & 0 \\ 14784 & 5280 & 1320 & 220 & 22 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 8^2 \\ 8^3 \\ 8^4 \\ 8^5 \end{pmatrix}.$$

The first column of the above matrix is A069720 in [28].

3.3. (a, b + (b + as), a) -colored Motzkin paths

The special generalized Chebyshev polynomials are defined as $G_k(z, s) = g_k(2z, 1, s)$, and the generating function is $\sum_{n=0}^{\infty} G_n(z, s)t^n = \frac{1-st}{1-2zt+t^2}$.

Theorem 3.6. Let $R_{n,k}$ be the number of (a, b + (b + as), a)-colored Motzkin paths ending at (n, k) and having weight $G_k(z, s)$. Then

$$\sum_{k=0}^{n} R_{n,k} G_k(z,s) = (2az+b)^n, \quad \forall n \in \mathbb{N},$$
(3.6)

that is, the total weight of the paths of length n is $(2az + b)^n$.

Proof. The last step of any path is one of step set $\{U, H, D\}$. Therefore, the numbers $R_{n,k}$ satisfy the following recurrence relations,

$$R_{n+1,0} = (b+as)R_{n,0} + aR_{n,1},$$

$$R_{n+1,k+1} = aR_{n,k} + bR_{n,k+1} + aR_{n,k+2}.$$

This implies that the infinite triangle $R = (R_{n,k})_{n,k \in \mathbb{N}}$ is a Riordan array and the generating functions for the *A*- and *Z*-sequences are $A(t) = a + bt + at^2$ and Z(t) = b + as + at, respectively. Using (1.3) and (1.7), we obtain

$$R^{-1} = \left(\frac{a - ast}{a + bt + at^2}, \frac{t}{a + bt + at^2}\right),$$

$$R = \left(\frac{2}{1 - (2as + b)t + \sqrt{(1 - bt)^2 - 4a^2t^2}}, \frac{1 - bt - \sqrt{(1 - bt)^2 - 4a^2t^2}}{2at}\right).$$

Thus, the identity (3.1) can now be written as $R * (G_k(z, s))_k = ((b + 2az)^n)_n$, which is equivalent to $R^{-1} * ((b + 2az)^k)_k = (G_n(z, s))_n$. This is easier to prove: $\left(\frac{a-ast}{a+bt+at^2}, \frac{t}{a+bt+at^2}\right) * \frac{1}{1-(b+2az)t} = \frac{a-ast}{a+bt+at^2} \cdot \frac{a+bt+at^2}{a+bt+at^2-(b+2az)t} = \frac{1-st}{1-2zt+t^2}$. \Box

For s = 1 - r and z = 1, we obtain Theorem 5.1 of Merlini and Sprugnoli [19]. For $s = 1 - \frac{r}{q}$ and z = 1, we obtain Theorem 5.4 of Merlini and Sprugnoli [19].

For a = 1, b = 2, s = -1, we obtain Theorem 4.2 of [19]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 & 0 \\ 14 & 28 & 20 & 7 & 1 & 0 \\ 42 & 90 & 75 & 35 & 9 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ W_1(z) \\ W_2(z) \\ W_4(z) \\ W_5(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 2z+2 \\ (2z+2)^2 \\ (2z+2)^3 \\ (2z+2)^4 \\ (2z+2)^5 \end{pmatrix},$$

where $(W_n(x))_n$ is the sequence of modified Chebyshev polynomials of the fourth kind.

Corollary 3.7. Let $R_{n,k}$ be the number of (a, b + (b + az), a)-colored Motzkin paths ending at (n, k) and having weight $T_k(z)$, where $\{T_k(z)\}_{k\geq 0}$ is the sequence of Chebyshev polynomials of the first kind with the generating function $\frac{1-zt}{1-2zt+t^2}$. Then

$$\sum_{k=0}^{n} R_{n,k} T_k(z) = (2az+b)^n, \quad \forall n \in \mathbb{N},$$
(3.7)

that is, the total weight of the paths of length n is $(2az + b)^n$.

Proof. The result follows by setting s = z in Theorem 3.6. \Box

Corollary 3.8. Let $R_{n,k}$ be the number of (a, b, a)-colored Motzkin paths ending at (n, k) and having weight $U_k(z)$. Then

$$\sum_{k=0}^{n} R_{n,k} U_k(z) = (2az+b)^n, \quad \forall n \in \mathbb{N},$$
(3.8)

that is, the total weight of the paths of length n is $(2az + b)^n$.

Proof. The result follows by setting s = 0 in Theorem 3.6. \Box

Although we know that the identity (3.8) is true, it would be nice to be able to write it explicitly. Fortunately, this can be achieved by first producing a closed formula for $R_{n,k}$. Since $R = \left(\frac{1-bt-\sqrt{(1-bt)^2-4a^2t^2}}{2a^2t^2}, \frac{1-bt-\sqrt{(1-bt)^2-4a^2t^2}}{2at}\right)$ and $R^{-1} = \left(\frac{a}{a+bt+at^2}, \frac{t}{a+bt+at^2}\right)$, by the Lagrange Inversion formula, we have

$$\begin{split} R_{n,k} &= [t^n] \frac{1 - bt - \sqrt{(1 - bt)^2 - 4a^2t^2}}{2a^2t^2} \left(\frac{1 - bt - \sqrt{(1 - bt)^2 - 4a^2t^2}}{2at} \right)^{k+1} \\ &= \frac{1}{a} [t^{n+1}] \left(\frac{1 - bt - \sqrt{(1 - bt)^2 - 4a^2t^2}}{2at} \right)^{k+1} \\ &= \frac{1}{a} [t^{n-k}] \frac{k + 1}{n+1} (a + bt + at^2)^{n+1} \\ &= [t^{n-k}] \frac{k+1}{n+1} \sum_{m=0}^{2n+2} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} {n+1 \choose j} {j \choose m-j} a^{n+m-2j} b^{2j-m} t^m \\ &= \frac{k+1}{n+1} \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} {n+1 \choose j} {j \choose n-k-j} a^{2n-k-2j} b^{2j-n+k}. \end{split}$$

Thus, we obtain the following explicit form of the identity:

$$\sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{k+1}{n+1} \binom{n+1}{j} \binom{j}{n-k-j} a^{2n-k-2j} b^{2j-n+k} U_k(z) = (2az+b)^n.$$
(3.9)

In terms of matrices, we have

$$(R_{i,j})_{i,j\geq 0} * (U_k(z))_k = ((2az + b)^n)_n.$$

Writing the first few rows of $(R_{i,j})_{i,j\geq 0}$, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ a^2 + b^2 & 2ab & a^2 & 0 & 0 \\ b^3 + 3a^2b & 3ab^2 + 2a^3 & 3a^2b & a^3 & 0 \\ b^4 + 6a^2b^2 + 2a^4 & 4ab^3 + 8a^2b & 6a^2b^2 + 3a^4 & 4a^3b & a^4 \end{pmatrix} \begin{pmatrix} 1 \\ U_1(z) \\ U_2(z) \\ U_3(z) \\ U_4(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 2az + b \\ (2az + b)^2 \\ (2az + b)^3 \\ (2az + b)^4 \end{pmatrix}$$

If z = 1, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ a^2 + b^2 & 2ab & a^2 & 0 & 0 \\ b^3 + 3a^2b & 3ab^2 + 2a^3 & 3a^2b & a^3 & 0 \\ b^4 + 6a^2b^2 + 2a^4 & 4ab^3 + 8a^2b & 6a^2b^2 + 3a^4 & 4a^3b & a^4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2a + b \\ (2a + b)^2 \\ (2a + b)^3 \\ (2a + b)^4 \end{pmatrix} .$$

If a = b = z = 1, then we obtain the identities (1.5) of Chen et al. [7],

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 \\ 9 & 12 & 9 & 4 & 1 & 0 \\ 21 & 30 & 25 & 14 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3^2 \\ 3^3 \\ 3^4 \\ 3^5 \end{pmatrix}$$

where the first column is the sequence of Motzkin numbers.

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