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Invasion speed of a predator–prey system

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ABSTRACT

This paper is concerned with the asymptotic spreading of a predator–prey system, which formulates that the predator invades the habitat of the prey. By constructing auxiliary equation, we obtain the speed of asymptotic spreading of the predator.

1. Introduction

In population dynamics, one important case is to model the recolonization by lupins of Mount St Helens' north slope (see Fagan and Bishop [1], Owen and Lewis [2]), which is often formulated by a predator–prey system. In the literature, one basic topic is to investigate the invasion process of the predator when the prey is the aboriginal while the predator is the invader. In the past three decades, much attention has been paid to the description of the process by traveling wave solutions.

To formulate the invasion process when the initial habitat size of the invader is finite, asymptotic spreading is another very useful index [3]. Of course, the asymptotic spreading and traveling wave solutions have relevance, e.g., in some systems, it has been proved that the speed of asymptotic spreading equals to the minimal wave speed of traveling wave solutions [3–7]. But in some reducible and nonmonotone systems, different species may have different speeds of asymptotic spreading while the minimal wave speed of a kind of traveling wave solutions is unique, e.g., the classical Lotka–Volterra competition system [8,9].

When the asymptotic spreading of predator–prey systems is concerned, Lin [10] and Pan [11] investigated the following predator–prey system,

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \Delta u_1(x,t) + u_1(x,t)\left[1 - u_1(x,t) - bu_2(x,t)\right], \\ \dfrac{\partial u_2(x,t)}{\partial t} = d\Delta u_2(x,t) + ru_2(x,t)\left[1 + fu_1(x,t) - u_2(x,t)\right], \end{cases} \tag{1.1}$$

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in which $x \in \mathbb{R}, t > 0$ and all the parameters are positive. They estimated the invasion speeds of u_1, u_2 when both u_1, u_2 are invaders, which may be different. It should be noted that the intrinsic growth rate of u_2 is also positive, and such an assumption is not precise in some biological processes. Moreover, Wang et al. [12] studied the spreading phenomena of (1.1) with free boundary and obtained a spreading–vanishing dichotomy.

In this paper, we investigate the following predator–prey system:

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \Delta u_1(x,t) + u_1(x,t)\left[1 - u_1(x,t) - bu_2(x,t)\right], \\ \dfrac{\partial u_2(x,t)}{\partial t} = d\Delta u_2(x,t) + ru_2(x,t)\left[-1 + fu_1(x,t) - u_2(x,t)\right], \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}, t > 0$ and all the parameters are positive. In Lin [13], the minimal wave speed of traveling wave solution is obtained, here a traveling wave solution formulates the predator invades the habitat of the prey and they coexist eventually, in which $2\sqrt{dr(f-1)}$ is the minimal wave speed. In what follows, we shall consider the similar biological process by the speed of asymptotic spreading.

From the viewpoint of monotone dynamical systems, the predator–prey system does not generate monotone semiflows. Therefore, some classical conclusions mentioned above cannot be applied. Furthermore, in [10,11], the positivity of the intrinsic growth rate plays an important role in constructing some auxiliary equations which admit comparison principle. In this paper, the negative intrinsic growth rate of the predator leads to the difficulty in constructing auxiliary equations similar to those in [10,11]. Therefore, some new techniques are necessary.

2. Main results

To estimate the invasion speed of u_2 in (1.2), we consider (1.2) with the following initial value condition:

$$u_1(x,0) = 1, u_2(x,0) = u(x), x \in \mathbb{R}, \tag{2.1}$$

where $u(x)$ is a continuous function admitting nonempty compact support and satisfying

$$0 \le u(x) \le f - 1, x \in \mathbb{R}.$$

Clearly, the above initial value condition implies that u_1 is the aboriginal while u_2 is the invader with finite size of initial habitat. The following is our main result.

Theorem 2.1. *Assume that $b(f-1) > 0$ holds. Let $c^* = 2\sqrt{dr(f-1)}$. If $(u_1(x,t), u_2(x,t))$ is defined by (1.2) with (2.1), then*

$$\lim_{t\to\infty, |x|<(c^*-\theta)t} u_2(x,t) = \frac{f-1}{bf+1}, \lim_{t\to\infty, |x|>(c^*+\theta)t} u_2(x,t) = 0 \tag{2.2}$$

for any $\theta \in (0, c^*)$.

Remark 2.2. This implies that c^* is the rough speed in which the predator invades the habitat of the prey. Moreover, if $b(f-1) > 0$, then (1.2) admits a spatially homogeneous positive steady state which is asymptotic stable.

Proof. By the basic theory of reaction–diffusion systems, we see

$$u_1(x,t) \in (0,1), u_2(x,t) \in (0, f-1), u_1(x,t) \in (a,1), x \in \mathbb{R}, t > 0,$$

where $a = 1 - b(f-1)$. Let $w_1 = 1 - u_1, w_2 = u_2$, then

$$\begin{cases} \dfrac{\partial w_1(x,t)}{\partial t} = \Delta w_1(x,t) + [1 - w_1(x,t)]\,[bw_2(x,t) - w_1(x,t)]\,, \\ \dfrac{\partial w_2(x,t)}{\partial t} = d\Delta w_2(x,t) + rw_2(x,t)[f - 1 - w_2(x,t) - fw_1(x,t)], \end{cases}$$

and

$$w_2(x,t) \in (0, f-1), w_1(x,t) \in (0, b(f-1)), 1 - w_1(x,t) \in (a,1), x \in \mathbb{R}, t > 0.$$

Moreover, by the classical theory of parabolic systems,

$$\frac{\partial w_1(x,t)}{\partial t}, \frac{\partial w_1(x,t)}{\partial x}, \frac{\partial w_2(x,t)}{\partial t}, \frac{\partial w_2(x,t)}{\partial x}$$

are uniformly bounded if $t > 1$, and we only consider $t > 1$ since the asymptotic spreading involves long time behavior.

For any given bounded and continuous function $v(x)$, we define

$$(T(t)v)(x) = \frac{e^{-at}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-y^2}{4t}} v(x-y)dy, t > 0$$

then $T(t), t > 0$ is an analytic semigroup.

From the positivity of $w_1(x,t), w_2(x,t)$,,

$$\begin{aligned} \frac{\partial w_1(x,t)}{\partial t} &= \Delta w_1(x,t) + bw_2(x,t) - bw_2(x,t)w_1(x,t) - w_1(x,t)[1 - w_1(x,t)] \\ &\le \Delta w_1(x,t) + bw_2(x,t) - aw_1(x,t). \end{aligned}$$

Utilizing the theory of semigroup, we have

$$\begin{aligned} w_1(x,t) &\le T(t)w_1(x,0) + b\int_0^t (T(t-s)w_2(s,\cdot))ds(x) \\ &= b\int_0^t (T(t-s)w_2(\cdot,s))ds(x) \end{aligned}$$

since $w_1(x,0) \equiv 0, x \in \mathbb{R}$.

With the estimation, $w_2(x,t)$ satisfies

$$\begin{aligned} \frac{\partial w_2(x,t)}{\partial t} &= d\Delta w_2(x,t) + rw_2(x,t)\,[f - 1 - fw_1(x,t) - w_2(x,t)] \\ &\ge d\Delta w_2(x,t) + rw_2(x,t)\,[f - 1 - A(x,t) - w_2(x,t)]\,, \end{aligned}$$

where

$$A(x,t) = fb\int_0^t (T(t-s)w_2(\cdot,s))ds(x)$$

and

$$\int_0^t (T(t-s)w_2(\cdot,s))ds(x) = \int_0^t e^{-a(t-s)} \left[\frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{\frac{-y^2}{4(t-s)}} w_2(x-y,s)dy\right] ds.$$

Note that $\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-y^2}{4t}} dy = 1$ for any $t > 0$ and w_2 is bounded and $\int_0^\infty e^{-as}ds = 1/a$, then for any $\epsilon > 0$ small enough, there exists $T = T(\epsilon) > 0$ such that

$$A(x,t) < \frac{\epsilon}{4} + bf\int_{t-T}^t (T(t-s)w_2(\cdot,s))ds(x)$$

for any $t > T + 1, x \in \mathbb{R}$. Further applying the uniform convergence, we see that

$$A(x,t) < \frac{\epsilon}{2} + bf \int_{t-T}^{t} e^{-a(t-s)} \left[\frac{1}{\sqrt{4\pi(t-s)}} \int_{-R}^{R} e^{\frac{-y^2}{4(t-s)}} w_2(x-y,s)dy \right] ds$$

for some $R > 0$ large enough (clearly, R is uniform in $t > T + 1, x \in \mathbb{R}$). With these estimations, $w_2(x,t)$ satisfies

$$\begin{aligned} \frac{\partial w_2(x,t)}{\partial t} &\geq d\Delta w_2(x,t) + r w_2(x,t) \\ &\times \left[f - 1 - \frac{\epsilon}{2} - bf \int_{t-T}^{t} e^{-a(t-s)} \left[\frac{1}{\sqrt{4\pi(t-s)}} \int_{-R}^{R} e^{\frac{-y^2}{4(t-s)}} w_2(x-y,s)dy \right] ds - w_2(x,t) \right] \end{aligned}$$

for any $t > T + 1, x \in \mathbb{R}$. Unfortunately, the above equation does not admit a comparison principle. To estimate the property of $w_2(x,t)$, we need further analysis.

If

$$bf \int_{t-T}^{t} e^{-a(t-s)} \left[\frac{1}{\sqrt{4\pi(t-s)}} \int_{-R}^{R} e^{\frac{-y^2}{4(t-s)}} w_2(x-y,s)dy \right] ds \leq \frac{\epsilon}{2},$$

then $w_2(x,t)$ satisfies

$$\frac{\partial w_2(x,t)}{\partial t} \geq d\Delta w_2(x,t) + r w_2(x,t) \left[f - 1 - \epsilon - w_2(x,t) \right].$$

Otherwise, let $s' = s'(\epsilon) > 0$ such that

$$bf(f-1) \int_{t-s'}^{t} e^{-a(t-s)} \left[\frac{1}{\sqrt{4\pi(t-s)}} \int_{-R}^{R} e^{\frac{-y^2}{4(t-s)}} dy \right] ds = \frac{\epsilon}{4}.$$

Then,

$$bf \int_{t-T}^{t-s'} e^{-a(t-s)} \left[\frac{1}{\sqrt{4\pi(t-s)}} \int_{-R}^{R} e^{\frac{-y^2}{4(t-s)}} w_2(x-y,s)dy \right] ds > \frac{\epsilon}{4},$$

and the uniform continuity of $w_2(x,t)$ implies that there exist a constant $\varepsilon = \varepsilon(\epsilon) > 0$ and

$$y \in [x-R, x+R], s \in [t-T, t-s'], t > T+1$$

such that

$$w_2(y,s) > 2\varepsilon.$$

Again, by the uniform continuity of $w_2(x,t)$, there exists $\delta \in (0,1]$ such that

$$w_2(z,s) > \varepsilon, -\delta < z - y < \delta.$$

Consider the following initial value problem:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = d\Delta w(x,t) + r w(x,t) \left[-1 - w(x,t) \right], \\ w(x,0) = w(x), \end{cases}$$

where $w(x)$ is a continuous function satisfying

(w1) $w(x) = \varepsilon, |x| \leq \frac{\delta}{2}$,
(w2) $w(x) = 0, |x| \geq \delta$,
(w3) if $x \in [\frac{\delta}{2}, \delta]$, then $w(x)$ is decreasing, if $x \in [-\delta, -\frac{\delta}{2}]$, then $w(x)$ is increasing.

Then, $w(x,t) > 0$ for any $x \in \mathbb{R}, t > 0$. Let

$$\eta = \min_{t\in[s',T],|x|<R+1} w(x,t),$$

then $\eta > 0$. What we have done implies that if

$$bf\int_{t-T}^{t} e^{-a(t-s)}\left[\frac{1}{\sqrt{4\pi(t-s)}}\int_{-R}^{R} e^{\frac{-y^2}{4(t-s)}} w_2(x-y,s)dy\right]ds > \frac{\epsilon}{2},$$

then $w_2(x,t) \geq \eta$ such that

$$A(x,t) \leq f \leq \frac{f}{\eta} w_2(x,t),$$

and so

$$\frac{\partial w_2(x,t)}{\partial t} \geq d\Delta w_2(x,t) + rw_2(x,t)\left[f-1-\epsilon-(1+f/\eta)w_2(x,t)\right]$$

for all $x \in \mathbb{R}, t > T+1$.

By the conclusion in [3], we obtain

$$\liminf_{t\to\infty} \inf_{|x|<ct} w_2(x,t) > 0$$

where $c < 2\sqrt{dr(f-1-\epsilon)}$. By the arbitrary of $\epsilon > 0$ and $u_2 = w_2$, we see that

$$\liminf_{t\to\infty} \inf_{|x|<(c^*-\theta)t} u_2(x,t) > 0$$

for any given $\theta \in (0,c^*)$, which further indicates that

$$\lim_{t\to\infty,|x|<(c^*-\theta)t} u_2(x,t) = \frac{f-1}{bf+1}$$

by the stability of the positive steady state.

Moreover, since

$$\frac{\partial w_2(x,t)}{\partial t} \leq d\Delta w_2(x,t) + w_2(x,t)[f-1-w_2(x,t)]$$

for $x \in \mathbb{R}, t > 0$. Then, [3] implies that

$$\lim_{t\to\infty,|x|>(c^*+\theta)t} u_2(x,t) = 0$$

for any given $\theta > 0$. The proof is complete. □

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