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Preconditioned HSS iteration method and its non-alternating variant for continuous Sylvester equations

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ABSTRACT

Bai (2010) proposed an efficient Hermitian and skew-Hermitian splitting (HSS) iteration method for solving a broad class of large sparse continuous Sylvester equations. To further improve the efficiency of the HSS method, in this paper we present a preconditioned HSS (PHSS) iteration method and its non-alternating variant (NPHSS) for this matrix equation. The convergence properties of the PHSS and NPHSS methods are studied in depth and the quasi-optimal values of the iteration parameters for the two methods are also derived. Moreover, to reduce the computational cost, we establish the inexact variants of the two iteration methods. Numerical experiments illustrate the efficiency and robustness of the two iteration methods and their inexact variants.

1. Introduction

We consider the following continuous Sylvester equations

$$AX + XB = C, \tag{1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$ are given complex matrices. Assume that

(i) A, B and C are large and sparse matrices;
(ii) at least one of A and B is non-Hermitian;
(iii) both A and B are positive semi-definite, and at least one of them is positive definite.

Since there is no common eigenvalue between A and $-B$, we can easily obtain that the continuous Sylvester equation (1) has a unique solution [1,2]. In addition, the continuous Lyapunov equation is a special case of the continuous Sylvester equation (1) with $B = A^*$ and C Hermitian, where A^* represents the conjugate transpose of the matrix A. Applications of this class of continuous Sylvester equations arise in several areas, such as control and system theory [3–6], stability of linear systems [7], linear algebra [8], signal processing [9], image restoration [10], filtering [11,12] and so on.

The above continuous Sylvester equation (1) can be equivalently rewritten as the following system of linear equations

$$\mathbf{A}x = c, \tag{2}$$

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where $\mathbf{A} = I_n \otimes A + B^T \otimes I_m$, the vectors x and c contain the concatenated columns of the matrices X and C, respectively, with I_m, I_n being the identity matrices of order m and n, respectively, $\otimes$ being the Kronecker product symbol and B^T representing the transpose of the matrix B. However, solving this equivalent linear system is quite expensive and ill-conditioned.

There are a large number of numerical methods for solving the continuous Sylvester equation (1). Direct algorithms can only be applied to problems of reasonably small size, such as the Bartels–Stewart and the Hessenberg–Schur methods [13,14]. Iterative methods are usually employed when the matrices A and B become large and sparse, for instance, the Smith's method [15], the alternating direction implicit (ADI) method [10,16–19], gradient based algorithm [20,21] and others [22–25]. We can also refer to [26] for a detailed survey of this area.

Recently, based on the Hermitian and skew-Hermitian splitting (HSS) of the matrices A and B, Bai proposed an HSS iteration method [27] for solving the continuous Sylvester equation (1). This HSS method is a matrix variant of the original HSS method for solving systems of linear equations, which is firstly presented by Bai, Golub and Ng [28]. Due to its promising performance and elegant convergence properties, many HSS-based methods for solving linear systems were subsequently studied to improve its robustness; see [29–41] and other literature. Hereafter, some of these methods were further considered for solving the above continuous Sylvester equation [42–48] and other linear matrix equations [49–51].

In this paper, we establish a preconditioned HSS (PHSS) iteration method and its non-alternating variant (NPHSS) for solving the continuous Sylvester equation (1). Both of the two methods are preconditioned iteration, which can improve convergence efficiency of the HSS iteration [27]. Similar approaches of using preconditioned technique in the algorithmic designs of the iterative methods can be seen in [32–34,39,52,53].

In the remainder of this paper, a matrix sequence $\{Y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ is said to be convergent to a matrix $Y \in \mathbb{C}^{m\times n}$ if the corresponding vector sequence $\{y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{mn}$ is convergent to the corresponding vector $y \in \mathbb{C}^{mn}$, where the vectors $y^{(k)}$ and y contain the concatenated columns of the matrices $Y^{(k)}$ and Y, respectively. If $\{Y^{(k)}\}_{k=0}^{\infty}$ is convergent, then its convergence factor and convergence rate are defined as those of $\{y^{(k)}\}_{k=0}^{\infty}$, correspondingly. In addition, we use $\|V\|_2$ and $\|V\|_F$ to denote the spectral norm and the Frobenius norm of the matrix $V \in \mathbb{C}^{m\times m}$, respectively. Note that $\|\cdot\|_2$ is also used to represent the 2-norm of a vector.

The rest of this paper is organized as follows. In Section 2, we present the PHSS method and its inexact variant for solving the continuous Sylvester equation (1), and the convergence theorems are studied. We establish the NPHSS method and its inexact variant for solving the continuous Sylvester equation (1) in Section 3, and the convergence theorems are also discussed. In Section 4, numerical results are given to illustrate the effectiveness of our two methods and their inexact variants. Finally, in Section 5, we end this work with a brief conclusion.

2. The preconditioned HSS (PHSS) iteration method

2.1. The PHSS method

Here and in the sequel, we use $H(V) := \frac{1}{2}(V + V^*)$ and $S(V) := \frac{1}{2}(V - V^*)$ to denote the Hermitian part and the skew-Hermitian part of a square matrix V, respectively. Obviously, the matrix V naturally possesses the Hermitian and skew-Hermitian splitting (HSS):

$$V = H(V) + S(V);$$

see [27,28]. Then we obtain the following splitting of A and B:

$$\begin{aligned} A &= (\alpha P_1 + H(A)) - (\alpha P_1 - S(A)) \\ &= (\alpha P_1 + S(A)) - (\alpha P_1 - H(A)), \end{aligned}$$

and

$$\begin{aligned} B &= (\alpha P_2 + H(B)) - (\alpha P_2 - S(B)) \\ &= (\alpha P_2 + S(B)) - (\alpha P_2 - H(B)), \end{aligned}$$

where α is a given positive constant and $P_1 \in \mathbb{C}^{m\times m}$, $P_2 \in \mathbb{C}^{n\times n}$ are two prescribed Hermitian positive definite matrices. Therefore, the continuous Sylvester equation (1) can be equivalently reformulated as

$$\begin{cases} (\alpha P_1 + H(A))X + X(\alpha P_2 + H(B)) = (\alpha P_1 - S(A))X + X(\alpha P_2 - S(B)) + C, \\ (\alpha P_1 + S(A))X + X(\alpha P_2 + S(B)) = (\alpha P_1 - H(A))X + X(\alpha P_2 - H(B)) + C. \end{cases} \tag{3}$$

Under the assumptions (i)–(iii), there is no common eigenvalue between the matrices $\alpha P_1 + H(A)$ and $-(\alpha P_2 + H(B))$, as well as between the matrices $\alpha P_1 + S(A)$ and $-(\alpha P_2 + S(B))$, so that the above two fixed-point matrix equations have unique solutions for all given right-hand side matrices. This leads to the following preconditioned HSS (PHSS) iteration method for solving the continuous Sylvester equation (1).

Algorithm 1 (*The PHSS Iteration Method*)**.** Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, compute $X^{(k+1)} \in \mathbb{C}^{m\times n}$ for $k = 0, 1, 2, \ldots$ using the following iteration scheme until $\{X^{(k)}\}_{k=0}^{\infty}$ satisfies the stopping criterion:

$$\begin{cases}(\alpha P_1 + H(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha P_2 + H(B)) = (\alpha P_1 - S(A))X^{(k)} + X^{(k)}(\alpha P_2 - S(B)) + C,\\ (\alpha P_1 + S(A))X^{(k+1)} + X^{(k+1)}(\alpha P_2 + S(B)) = (\alpha P_1 - H(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\alpha P_2 - H(B)) + C\end{cases} \tag{4}$$

where α is a given positive constant and $P_1 \in \mathbb{C}^{m\times m}$, $P_2 \in \mathbb{C}^{n\times n}$ are two prescribed Hermitian positive definite matrices.

Remark 1. It is easy to see that the PHSS iteration method reduces to the HSS iteration method [27] with $P_1 = I_m, P_2 = \frac{\beta}{\alpha} I_n$, where β is another given positive constant.

Remark 2. When B is a zero matrix, and $X^{(k)}$ and C reduce to column vectors, the PHSS iteration method becomes the one for systems of linear equations; see [32,33]. In addition, when $B = A^*$ and C is Hermitian, it leads to a PHSS iteration method for the continuous Lyapunov equations.

2.2. Convergence analysis of the PHSS method

Denote by $\mathbf{A} = \mathbf{H} + \mathbf{S}$, with

$$\mathbf{H} = H(\mathbf{A}) = I_n \otimes H(A) + H(B)^T \otimes I_m \quad \text{and} \quad \mathbf{S} = S(\mathbf{A}) = I_n \otimes S(A) + S(B)^T \otimes I_m, \tag{5}$$

and

$$\mathbf{P} = I_n \otimes P_1 + P_2^T \otimes I_m.$$

We can easily verify that $\mathbf{H}$ is a Hermitian positive definite matrix, $\mathbf{S}$ is a skew-Hermitian matrix, and $\mathbf{P}$ is a Hermitian positive definite matrix. Therefore, all eigenvalues of $\mathbf{P}^{-1}\mathbf{H}$ are real and positive, and all eigenvalues of $\mathbf{P}^{-1}\mathbf{S}$ are imaginary. Here and in the sequel, denote

$$\Lambda_{\max} = \max_{\Lambda_j \in \mathrm{sp}(\mathbf{P}^{-1}\mathbf{H})} \{\Lambda_j\}, \quad \Lambda_{\min} = \min_{\Lambda_j \in \mathrm{sp}(\mathbf{P}^{-1}\mathbf{H})} \{\Lambda_j\} \text{ and } \Xi_{\max} = \max_{i\Xi_j \in \mathrm{sp}(\mathbf{P}^{-1}\mathbf{S})} \{|\Xi_j|\},$$

where $\mathrm{sp}(V)$ denotes the spectrum of the matrix V and $i = \sqrt{-1}$.

The following theorem gives the convergence results of the PHSS iteration method for solving the continuous Sylvester equation (1).

Theorem 1. *Assume that $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are positive semi-definite matrices, and at least one of them is positive definite. Denote by*

$$M(\alpha, \mathbf{P}) = (\alpha\mathbf{P} + \mathbf{S})^{-1}(\alpha\mathbf{P} - \mathbf{H})(\alpha\mathbf{P} + \mathbf{H})^{-1}(\alpha\mathbf{P} - \mathbf{S}). \tag{6}$$

Then the convergence factor of the PHSS iteration method (4) is given by the spectral radius $\rho(M(\alpha, \mathbf{P}))$ of the matrix $M(\alpha, \mathbf{P})$, which is bounded by

$$\sigma(\alpha, \mathbf{P}) := \max_{\Lambda_j \in sp(\mathbf{P}^{-1}\mathbf{H})} \frac{|\alpha - \Lambda_j|}{|\alpha + \Lambda_j|}. \tag{7}$$

Consequently, we have

$$\rho(M(\alpha, \mathbf{P})) \le \sigma(\alpha, \mathbf{P}) < 1, \qquad \forall \alpha > 0, \tag{8}$$

i.e., the PHSS iteration method (4) is unconditionally convergent to the exact solution $X^\star \in \mathbb{C}^{m\times n}$ of the continuous Sylvester equation (1).

Moreover, the minimum point $\alpha^\star$ and the minimum value $\sigma(\alpha^\star, \mathbf{P})$ of the upper bound $\sigma(\alpha, \mathbf{P})$ are respectively as

$$\alpha^\star \equiv \arg\min_{\alpha} \{\sigma(\alpha, \mathbf{P})\} = \arg\min_{\alpha} \left\{ \max_{\Lambda_j \in sp(\mathbf{P}^{-1}\mathbf{H})} \left| \frac{\alpha - \Lambda_j}{\alpha + \Lambda_j} \right| \right\} = \sqrt{\Lambda_{\min}\Lambda_{\max}} \tag{9}$$

and

$$\sigma(\alpha^\star, \mathbf{P}) = \frac{\sqrt{\Lambda_{\max}} - \sqrt{\Lambda_{\min}}}{\sqrt{\Lambda_{\max}} + \sqrt{\Lambda_{\min}}}. \tag{10}$$

Proof. By making use of the Kronecker product, we can reformulate the PHSS iteration (4) as the following matrix–vector form:

$$\begin{cases}(I_n \otimes (\alpha P_1 + H(A)) + (\alpha P_2 + H(B))^T \otimes I_m)x^{(k+\frac{1}{2})} \\ \qquad = (I_n \otimes (\alpha P_1 - S(A)) + (\alpha P_2 - S(B))^T \otimes I_m)x^{(k)} + c, \\ (I_n \otimes (\alpha P_1 + S(A)) + (\alpha P_2 + S(B))^T \otimes I_m)x^{(k+1)} \\ \qquad = (I_n \otimes (\alpha P_1 - H(A)) + (\alpha P_2 - H(B))^T \otimes I_m)x^{(k+\frac{1}{2})} + c,\end{cases}$$

which can be arranged equivalently as

$$\begin{cases}(\alpha \mathbf{P} + \mathbf{H})x^{(k+\frac{1}{2})} = (\alpha \mathbf{P} - \mathbf{S})x^{(k)} + c, \\ (\alpha \mathbf{P} + \mathbf{S})x^{(k+1)} = (\alpha \mathbf{P} - \mathbf{H})x^{(k+\frac{1}{2})} + c.\end{cases} \tag{11}$$

Evidently, the iteration scheme (11) is the PHSS iteration method for solving the system of linear equations (2), with $\mathbf{A} = \mathbf{H} + \mathbf{S}$; see [33]. After concrete operations, the PHSS iteration (11) can be also expressed as a stationary iteration as follows:

$$x^{(k+1)} = M(\alpha, \mathbf{P})x^{(k)} + N(\alpha, \mathbf{P})c,$$

where $M(\alpha, \mathbf{P})$ is the iteration matrix defined in (6), and

$$N(\alpha, \mathbf{P}) = 2\alpha(\alpha \mathbf{P} + \mathbf{S})^{-1}(\alpha \mathbf{P} + \mathbf{H})^{-1}.$$

Hence, the conclusion is straightforward according to Theorem 2.1 in [33]. This completes the proof. □

2.3. Inexact PHSS iteration method

In the process of PHSS iteration (4), two sub-problems need to be solved exactly. To further improve computational efficiency of the PHSS iteration, we establish an inexact PHSS (IPHSS) iteration, which solves the two sub-problems iteratively [10,15–18,22,23]. We write the IPHSS iteration scheme in the following algorithm for solving the continuous Sylvester equation (1).

Algorithm 2 (*The IPHSS Iteration Method*). Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, then this algorithm leads to the solution of the continuous Sylvester equation (1):

$k = 0$;
while (not convergent)
$R^{(k)} = C - AX^{(k)} - X^{(k)}B$;
approximately solve $(\alpha P_1 + H(A))Z^{(k)} + Z^{(k)}(\alpha P_2 + H(B)) = R^{(k)}$ by employing an effective iteration method, such that the residual $P^{(k)} = R^{(k)} - (\alpha P_1 + H(A))Z^{(k)} - Z^{(k)}(\alpha P_2 + H(B))$ of the iteration satisfies $\|P^{(k)}\|_F \le \varepsilon_k \|R^{(k)}\|_F$;
$X^{(k+\frac{1}{2})} = X^{(k)} + Z^{(k)}$;
$R^{(k+\frac{1}{2})} = C - AX^{(k+\frac{1}{2})} - X^{(k+\frac{1}{2})}B$;
approximately solve $(\alpha P_1 + S(A))Z^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}(\alpha P_2 + S(B)) = R^{(k+\frac{1}{2})}$ by employing an effective iteration method, such that the residual $Q^{(k+\frac{1}{2})} = R^{(k+\frac{1}{2})} - (\alpha P_1 + S(A))Z^{(k+\frac{1}{2})} - Z^{(k+\frac{1}{2})}(\alpha P_2 + S(B))$ of the iteration satisfies $\|Q^{(k+\frac{1}{2})}\|_F \le \eta_k \|R^{(k+\frac{1}{2})}\|_F$;
$X^{(k+1)} = X^{(k+\frac{1}{2})} + Z^{(k+\frac{1}{2})}$;
$k = k + 1$;
end.

Here, $\{\varepsilon_k\}$ and $\{\eta_k\}$ are prescribed tolerances used to control the accuracies of the inner iterations.

We remark that when $P_1 = I_m$ and $P_2 = \frac{\beta}{\alpha} I_n$, the IPHSS method reduces to the inexact HSS (IHSS) method [27] for solving the continuous Sylvester equation (1).

The convergence properties for the two-step iteration have been carefully studied in [28,31]. By making use of Theorem 3.1 in [28], we can demonstrate the following convergence results about the above IPHSS iteration method.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. If $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ is an iteration sequence generated by the IPHSS iteration method and if $X^\star \in \mathbb{C}^{m\times n}$ is the exact solution of the continuous Sylvester equation (1), then it holds that*

$$\|X^{(k+1)} - X^\star\|_S \le (\sigma(\alpha, \mathbf{P}) + \theta\nu\eta_k)(1 + \theta\varepsilon_k)\|X^{(k)} - X^\star\|_S, \quad k = 0, 1, 2, \ldots$$

where the norm $\|\cdot\|_S$ is defined as

$$\|Y\|_S = \|(\alpha P_1 + S(A))Y + Y(\alpha P_2 + S(B))\|_F$$

for any matrix $Y \in \mathbb{C}^{m\times n}$, and the constants θ and ν are given by

$$\theta = \|\mathbf{A}(\alpha \mathbf{P} + \mathbf{S})^{-1}\|_2, \quad \nu = \|(\alpha \mathbf{P} + \mathbf{S})(\alpha \mathbf{P} + \mathbf{H})^{-1}\|_2. \tag{12}$$

In particular, when

$$(\sigma(\alpha, \mathbf{P}) + \theta\nu\eta_{\max})(1 + \theta\varepsilon_{\max}) < 1, \tag{13}$$

the iteration sequence $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ *converges to* $X^\star \in \mathbb{C}^{m\times n}$, *where* $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$ *and* $\eta_{\max} = \max_k\{\eta_k\}$.

Proof. By making use of the Kronecker product and the notations introduced in Section 2.2, we can reformulate the above-described IPHSS iteration as the following matrix–vector form:

$$\begin{cases} (\alpha\mathbf{P} + \mathbf{H})z^{(k)} = r^{(k)}, & x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}, \\ (\alpha\mathbf{P} + \mathbf{S})z^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})}, & x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}, \end{cases} \tag{14}$$

with $r^{(k)} = c - \mathbf{A}x^{(k)}$ and $r^{(k+\frac{1}{2})} = c - \mathbf{A}x^{(k+\frac{1}{2})}$, where $z^{(k)}$ is such that the residual

$$p^{(k)} = r^{(k)} - (\alpha\mathbf{P} + \mathbf{H})z^{(k)}$$

satisfies $\|p^{(k)}\|_2 \le \varepsilon_k\|r^{(k)}\|_2$, and $z^{(k+\frac{1}{2})}$ is such that the residual

$$q^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})} - (\alpha\mathbf{P} + \mathbf{S})z^{(k+\frac{1}{2})}$$

satisfies $\|q^{(k+\frac{1}{2})}\|_2 \le \eta_k\|r^{(k+\frac{1}{2})}\|_2$.

Evidently, the iteration scheme (14) is the inexact PHSS iteration method for solving the system of linear equations (2), with $\mathbf{A} = \mathbf{H} + \mathbf{S}$. Then, by making use of Theorem 3.1 in [28] we can obtain the estimate

$$|||x^{(k+1)} - x^\star||| \le (\sigma(\alpha, \mathbf{P}) + \mu\theta\varepsilon_k + \theta(\rho + \theta\nu\varepsilon_k)\eta_k)|||x^{(k)} - x^\star|||, \quad k = 0, 1, 2, \ldots \tag{15}$$

where the constants θ and ν are defined in (12), μ and ρ are given by

$$\mu = \|(\alpha\mathbf{P} - \mathbf{H})(\alpha\mathbf{P} + \mathbf{H})^{-1}\|_2, \ \ \rho = \|(\alpha\mathbf{P} + \mathbf{S})(\alpha\mathbf{P} + \mathbf{H})^{-1}(\alpha\mathbf{P} - \mathbf{S})(\alpha\mathbf{P} + \mathbf{S})^{-1}\|_2,$$

and the norm $|||\cdot|||$ is defined as follows: for a vector $y \in \mathbb{C}^{mn}$, $|||y||| = \|(\alpha\mathbf{P} + \mathbf{S})y\|_2$. We can easily obtain that

$$\mu \le \sigma(\alpha, \mathbf{P}), \quad \rho \le \nu,$$

and for a matrix $Y \in \mathbb{C}^{m\times n}$,

$$|||y||| = \|(\alpha\mathbf{P} + \mathbf{S})y\|_2 = \|(\alpha P_1 + S(A))Y + Y(\alpha P_2 + S(B))\|_F = \|Y\|_S,$$

where the vector y contains the concatenated columns of the matrix Y.

Hence, we can equivalently rewrite the estimate (15) as

$$\|X^{(k+1)} - X^\star\|_S \le (\sigma(\alpha, \mathbf{P}) + \theta\nu\eta_k)(1 + \theta\varepsilon_k)\|X^{(k)} - X^\star\|_S, \quad k = 0, 1, 2, \ldots.$$

This proves the theorem. □

Remark 3. Theorem 2 shows that in order to guarantee the convergence of the IPHSS iteration, it is not necessary for $\{\varepsilon_k\}$ and $\{\eta_k\}$ to approach to zero as k is increasing. All we need is that the condition (13) is satisfied.

3. The non-alternating preconditioned HSS (NPHSS) iteration method

3.1. The NPHSS method

Different from the above two fixed-point matrix equations (3), the continuous Sylvester equation (1) can be equivalently reformulated as following non-alternating form

$$(\alpha P_1 + H(A))X + X(\alpha P_2 + H(B)) = (\alpha P_1 - S(A))X + X(\alpha P_2 - S(B)) + C. \tag{16}$$

Similarly, the above fixed-point matrix equation has a unique solution for all given right-hand side matrices. This leads to the following non-alternating preconditioned HSS (NPHSS) iteration method for solving the continuous Sylvester equation (1).

Algorithm 3 (*The NPHSS Iteration Method*)**.** Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, compute $X^{(k+1)} \in \mathbb{C}^{m\times n}$ for $k = 0, 1, 2, \ldots$ using the following iteration scheme until $\{X^{(k)}\}_{k=0}^{\infty}$ satisfies the stopping criterion:

$$(\alpha P_1 + H(A))X^{(k+1)} + X^{(k+1)}(\alpha P_2 + H(B)) = (\alpha P_1 - S(A))X^{(k)} + X^{(k)}(\alpha P_2 - S(B)) + C \tag{17}$$

where α is a given positive constant and $P_1 \in \mathbb{C}^{m\times m}$, $P_2 \in \mathbb{C}^{n\times n}$ are two prescribed Hermitian positive definite matrices.

Due to Hermitian positive definiteness of the matrices $\alpha P_1 + H(A)$ and $\alpha P_2 + H(B)$, every sub-system in (17) can be effectively solved.

Remark 4. When we take $P_1 = I_m$ and $P_2 = \frac{\beta}{\alpha} I_n$, the NPHSS iteration method reduces to a non-alternating HSS (NHSS) iteration method for solving the continuous Sylvester equation (1).

Remark 5. When B is a zero matrix, and $X^{(k)}$ and C reduce to column vectors, the NPHSS iteration method becomes the one for systems of linear equations [34], and the NHSS iteration method becomes the one for systems of linear equations [34,35]. In addition, when $B = A^*$ and C is Hermitian, it leads to an NPHSS iteration method for the continuous Lyapunov equations.

3.2. Convergence analysis of the NPHSS method

By making use of Theorem 1 and Corollary 1 in [34], we can obtain the following convergence theorem about the NPHSS iteration method for solving the continuous Sylvester equation (1).

Theorem 3. *Assume that $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are positive semi-definite matrices, and at least one of them is positive definite. Denote by*

$$\bar{M}(\alpha, \mathbf{P}) = (\alpha \mathbf{P} + \mathbf{H})^{-1}(\alpha \mathbf{P} - \mathbf{S}). \tag{18}$$

Then the convergence factor of the NPHSS iteration method (17) is given by the spectral radius $\rho(\bar{M}(\alpha, \mathbf{P}))$ of the matrix $\bar{M}(\alpha, \mathbf{P})$, which is bounded by

$$\bar{\sigma}(\alpha, \mathbf{P}) := \frac{\sqrt{\alpha^2 + \Xi_{\max}^2}}{\alpha + \Lambda_{\min}}. \tag{19}$$

Consequently, it holds that

(i) *If $\Lambda_{\min} \geq \Xi_{\max}$, then $\bar{\sigma}(\alpha, \mathbf{P}) < 1$ for any $\alpha > 0$, which means that the NPHSS iteration method is unconditionally convergent;*
(ii) *if $\Lambda_{\min} < \Xi_{\max}$, then $\bar{\sigma}(\alpha, \mathbf{P}) < 1$ if and only if*

$$\alpha > \frac{\Xi_{\max}^2 - \Lambda_{\min}^2}{2\Lambda_{\min}}, \tag{20}$$

which means that the NPHSS iteration method is convergent under the condition (20).

Moreover, the minimum point $\bar{\alpha}^\star$ and the minimum value $\bar{\sigma}(\bar{\alpha}^\star, \mathbf{P})$ of the upper bound $\bar{\sigma}(\alpha, \mathbf{P})$ are respectively as

$$\bar{\alpha}^\star \equiv \arg\min_{\alpha} \{\bar{\sigma}(\alpha, \mathbf{P})\} = \arg\min_{\alpha} \left\{ \frac{\sqrt{\alpha^2 + \Xi_{\max}^2}}{\alpha + \Lambda_{\min}} \right\} = \frac{\Xi_{\max}^2}{\Lambda_{\min}} \tag{21}$$

and

$$\bar{\sigma}(\bar{\alpha}^\star, \mathbf{P}) = \frac{\Xi_{\max}}{\sqrt{\Lambda_{\min}^2 + \Xi_{\max}^2}}. \tag{22}$$

Proof. By making use of the Kronecker product and the notations introduced in Section 2.2, we can reformulate the NPHSS iteration (17) as the following matrix–vector form:

$$(I_n \otimes (\alpha P_1 + H(A)) + (\alpha P_2 + H(B))^T \otimes I_m)x^{(k+1)}$$
$$= (I_n \otimes (\alpha P_1 - S(A)) + (\alpha P_2 - S(B))^T \otimes I_m)x^{(k)} + c,$$

which can be arranged equivalently as

$$(\alpha \mathbf{P} + \mathbf{H})x^{(k+1)} = (\alpha \mathbf{P} - \mathbf{S})x^{(k)} + c. \tag{23}$$

Evidently, the iteration scheme (23) is the NPHSS iteration method for solving the system of linear equations (2), with $\mathbf{A} = \mathbf{H} + \mathbf{S}$; see [34]. After concrete operations, the NPHSS iteration (23) can be also expressed as a stationary iteration as follows:

$$x^{(k+1)} = \bar{M}(\alpha, \mathbf{P})x^{(k)} + \bar{N}(\alpha, \mathbf{P})c,$$

where $\bar{M}(\alpha, \mathbf{P})$ is the iteration matrix defined in (18), and

$$\bar{N}(\alpha, \mathbf{P}) = (\alpha \mathbf{P} + \mathbf{H})^{-1}.$$

Hence, the conclusion is straightforward according to Theorem 1 and Corollary 1 in [34]. This completes the proof. □

3.3. Inexact NPHSS iteration method

Similar to inexact PHSS (IPHSS) iteration, we can also develop an inexact NPHSS (INPHSS) iteration for solving the continuous Sylvester equation (1). We write the INPHSS iteration scheme in the following algorithm.

Algorithm 4 (*The INPHSS Iteration Method*)**.** Given an initial guess $X^{(0)} \in \mathbb{C}^{m\times n}$, then this algorithm leads to the solution of the continuous Sylvester equation (1):

$k = 0$;
while (not convergent)
$R^{(k)} = C - AX^{(k)} - X^{(k)}B$;
approximately solve $(\alpha P_1 + H(A))Z^{(k)} + Z^{(k)}(\alpha P_2 + H(B)) = R^{(k)}$ by employing an effective iteration method, such that the residual $P^{(k)} = R^{(k)} - (\alpha P_1 + H(A))Z^{(k)} - Z^{(k)}(\alpha P_2 + H(B))$ of the iteration satisfies $\|P^{(k)}\|_F \le \varepsilon_k \|R^{(k)}\|_F$;
$X^{(k+1)} = X^{(k)} + Z^{(k)}$;
$k = k + 1$;
end.

Here, $\{\varepsilon_k\}$ is a prescribed tolerance used to control the accuracies of the inner iterations.

We remark that when $P_1 = I_m$ and $P_2 = \frac{\beta}{\alpha} I_n$, the IPHSS method reduces to the inexact NHSS (INHSS) method for solving the continuous Sylvester equation (1).

By making use of Theorem 3 in [34], we can demonstrate the following convergence results about the above INPHSS iteration method.

Theorem 4. *Let the conditions of Theorem 3 be satisfied. If $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ is an iteration sequence generated by the INPHSS iteration method and if $X^\star \in \mathbb{C}^{m\times n}$ is the exact solution of the continuous Sylvester equation (1), then it holds that*

$$\|X^{(k+1)} - X^\star\|_F \le (\bar{\sigma}(\alpha, \mathbf{P}) + \bar{\mu}\bar{\theta}\varepsilon_k)\|X^{(k)} - X^\star\|_F, \quad k = 0, 1, 2, \ldots$$

where the constants $\bar{\mu}$ and $\bar{\theta}$ are given by

$$\bar{\mu} = \|(\alpha\mathbf{P} + \mathbf{H})^{-1}\|_2, \quad \bar{\theta} = \|\mathbf{A}\|_2.$$

In particular, when

$$\bar{\sigma}(\alpha, \mathbf{P}) + \bar{\mu}\bar{\theta}\varepsilon_{\max} < 1, \tag{24}$$

the iteration sequence $\{X^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{m\times n}$ converges to $X^\star \in \mathbb{C}^{m\times n}$, where $\varepsilon_{\max} = \max_k\{\varepsilon_k\}$.

Proof. By making use of the Kronecker product and the notations introduced in Section 2.2, we can reformulate the above-described INPHSS iteration as the following matrix–vector form:

$$(\alpha\mathbf{P} + \mathbf{H})z^{(k)} = r^{(k)}, \qquad x^{(k+1)} = x^{(k)} + z^{(k)}, \tag{25}$$

with $r^{(k)} = c - \mathbf{A}x^{(k)}$, where $z^{(k)}$ is such that the residual

$$p^{(k)} = r^{(k)} - (\alpha\mathbf{P} + \mathbf{H})z^{(k)}$$

satisfies $\|p^{(k)}\|_2 \le \varepsilon_k \|r^{(k)}\|_2$.

Evidently, the iteration scheme (25) is the inexact NPHSS iteration method for solving the system of linear equations (2), with $\mathbf{A} = \mathbf{H} + \mathbf{S}$; see [34]. Then, by making use of Theorem 3 in [34] we can obtain the estimate

$$\|x^{(k+1)} - x^\star\|_2 \le (\bar{\sigma}(\alpha, \mathbf{P}) + \bar{\mu}\bar{\theta}\varepsilon_k)\|x^{(k)} - x^\star\|_2, \quad k = 0, 1, 2, \ldots. \tag{26}$$

Then we can equivalently rewrite the estimate (26) as

$$\|X^{(k+1)} - X^\star\|_F \le (\bar{\sigma}(\alpha, \mathbf{P}) + \bar{\mu}\bar{\theta}\varepsilon_k)\|X^{(k)} - X^\star\|_F, \quad k = 0, 1, 2, \ldots.$$

This proves the theorem. □

4. Numerical results

In this section, we are going to examine the feasibility and efficiency of PHSS, NHSS and NPHSS iteration methods and their inexact variants for solving the continuous Sylvester equation (1). In the following, the efficiency of all the iteration methods is tested by comparing the number of iteration steps (denoted as IT) and the computing time (in seconds, denoted as CPU). The numerical experiments are performed in Matlab on an Intel(R) Core(TM) i5 processor (2.40 GHz, 8GB RAM).

Table 1
The experimental optimal iteration parameters for HSS, PHSS, NHSS and NPHSS.

Method		HSS	PHSS	NHSS	NPHSS
$n = 10$	$q = 0.05$	2.65	0.70	0.01	0.01
	$q = 0.1$	2.77	0.73	0.01	0.01
	$q = 0.2$	2.89	0.76	0.08	0.03
	$q = 0.5$	3.00	0.79	3.81	1.00
	$q = 1$	2.75	0.87	15.9	4.17
$n = 20$	$q = 0.05$	1.99	0.62	0.01	0.01
	$q = 0.1$	2.10	0.65	0.01	0.01
	$q = 0.2$	2.19	0.68	0.28	0.09
	$q = 0.5$	2.26	0.70	4.69	1.46
	$q = 1$	2.45	0.76	19.9	6.62
$n = 40$	$q = 0.05$	1.85	0.61	0.01	0.01
	$q = 0.1$	1.83	0.60	0.01	0.01
	$q = 0.2$	1.92	0.63	0.29	0.10
	$q = 0.5$	1.91	0.63	5.25	1.72
	$q = 1$	1.96	0.64	23.5	7.70
$n = 80$	$q = 0.05$	1.73	0.58	0.01	0.01
	$q = 0.1$	1.73	0.58	0.02	0.01
	$q = 0.2$	1.81	0.60	0.27	0.09
	$q = 0.5$	1.82	0.61	4.86	1.62
	$q = 1$	1.83	0.61	22.0	7.28
$n = 160$	$q = 0.05$	1.63	0.54	0.01	0.01
	$q = 0.1$	1.76	0.59	0.02	0.02
	$q = 0.2$	1.72	0.58	0.44	0.15
	$q = 0.5$	1.74	0.58	4.87	1.63
	$q = 1$	1.68	0.56	21.0	6.97

In our implementations, the initial guess is chosen to be the zero matrix, and the iteration is terminated once the current iterate $X^{(k)}$ satisfies

$$\frac{\|C - AX^{(k)} - X^{(k)}B\|_F}{\|C\|_F} \leq 10^{-6}.$$

In addition, all the sub-problems involved in each step of the HSS, PHSS, NHSS and NPHSS iteration methods are solved exactly by the Bartels–Stewart method in [13]. In IHSS, IPHSS, INHSS and INPHSS iteration methods, we set $\varepsilon_k = \eta_k = 0.01$, $k = 0, 1, 2, \ldots$, and use the Smith's method [15] as the inner iteration scheme.

We consider the continuous Sylvester equation (1) with $m = n$ and the matrices

$$A = M_A + 2qN_A + \frac{100}{(n+1)^2}I, \quad B = M_B + 2qN_B + \frac{100}{(n+1)^2}I,$$

where $M_A, N_A, M_B, N_B \in \mathbb{R}^{n\times n}$ are the tridiagonal matrices given by

$$M_A = \text{tridiag}(-1, 2, -1), \quad N_A = \text{tridiag}(1.5, 0, -1.5),$$
$$M_B = \text{tridiag}(-1, 4, -1), \quad N_B = \text{tridiag}(3, 0, -3);$$

see also in [27].

For convenience, the preconditioners P_1 and P_2 involved in the PHSS and NPHSS methods are chosen to be the diagonal parts of the coefficient matrices A and B, respectively. For the actual iteration parameters of the HSS and NHSS methods, we take $\alpha = \beta$.

In Tables 1–2, for various problem sizes n and parameters q, we list the experimental optimal iteration parameters and the theoretical quasi-optimal iteration parameters of the four iteration methods, respectively. From Tables 1–2, we can see that when n increases, the optimal iteration parameters of the NHSS and NPHSS methods are increasing as well, while those of the HSS and PHSS methods are gradually decreasing. Moreover, for all cases, the optimal iteration parameters in the preconditioned methods are much smaller than those in the corresponding non-preconditioned methods.

In Tables 3–4, numerical results for HSS, PHSS, NHSS and NPHSS with the experimental optimal iteration parameters and the theoretical quasi-optimal iteration parameters are listed, respectively. From Tables 3–4, we can observe that the PHSS method is superior to the HSS method both in terms of iteration step and CPU time, and the NPHSS method considerably outperforms the NHSS method both in iteration step and CPU time. We also see that when q is small (the Hermitian part $\mathbf{H}$ is dominant), NHSS and NPHSS methods, no matter compared with the experimental optimal parameters or compared with the theoretical quasi-optimal parameters, perform much better than HSS and PHSS methods both in iteration step

Table 2
The theoretical quasi-optimal iteration parameters for HSS, PHSS, NHSS and NPHSS.

Method		HSS	PHSS	NHSS	NPHSS
$n=10$	$q=0.05$	3.3105	0.8652	0.0977	0.0128
	$q=0.1$	3.3105	0.8652	0.3909	0.0511
	$q=0.2$	3.3105	0.8652	1.5638	0.2043
	$q=0.5$	3.3105	0.8652	9.7736	1.2771
	$q=1$	3.3105	0.8652	39.0943	5.1084
$n=20$	$q=0.05$	2.5497	0.7902	0.1585	0.0246
	$q=0.1$	2.5497	0.7902	0.6341	0.0983
	$q=0.2$	2.5497	0.7902	2.5363	0.3930
	$q=0.5$	2.5497	0.7902	15.8516	2.4563
	$q=1$	2.5497	0.7902	63.4064	9.8251
$n=40$	$q=0.05$	2.3203	0.7584	0.1890	0.0309
	$q=0.1$	2.3203	0.7584	0.7559	0.1235
	$q=0.2$	2.3203	0.7584	3.0234	0.4941
	$q=0.5$	2.3203	0.7584	18.8963	3.0882
	$q=1$	2.3203	0.7584	75.5853	12.3526
$n=80$	$q=0.05$	2.2578	0.7488	0.1989	0.0330
	$q=0.1$	2.2578	0.7488	0.7955	0.1319
	$q=0.2$	2.2578	0.7488	3.1818	0.5276
	$q=0.5$	2.2578	0.7488	19.8865	3.2977
	$q=1$	2.2578	0.7488	79.5462	13.1907
$n=160$	$q=0.05$	2.2416	0.7462	0.2016	0.0336
	$q=0.1$	2.2416	0.7462	0.8063	0.1342
	$q=0.2$	2.2416	0.7462	3.2251	0.5368
	$q=0.5$	2.2416	0.7462	20.1569	3.3552
	$q=1$	2.2416	0.7462	80.6274	13.4206

Table 3
Numerical results for HSS, PHSS, NHSS and NPHSS with the experimental optimal iteration parameters.

Method	n	HSS		PHSS		NHSS		NPHSS	
q		IT	CPU	IT	CPU	IT	CPU	IT	CPU
$n=10$	$q=0.05$	9	0.0040	7	0.0034	3	0.0012	3	0.0011
	$q=0.1$	9	0.0043	7	0.0040	4	0.0016	3	0.0015
	$q=0.2$	9	0.0045	7	0.0039	7	0.0030	6	0.0025
	$q=0.5$	8	0.0037	6	0.0035	22	0.0122	18	0.0085
	$q=1$	9	0.0054	7	0.0038	66	0.0258	52	0.0179
$n=20$	$q=0.05$	11	0.0096	9	0.0135	3	0.0043	3	0.0034
	$q=0.1$	11	0.0154	9	0.0140	5	0.0049	4	0.0041
	$q=0.2$	11	0.0105	9	0.0085	9	0.0075	8	0.0071
	$q=0.5$	10	0.0128	8	0.0118	36	0.0386	31	0.0305
	$q=1$	10	0.0155	8	0.0133	116	0.0900	101	0.0704
$n=40$	$q=0.05$	11	0.0630	8	0.0267	3	0.0145	3	0.0138
	$q=0.1$	12	0.0401	9	0.0300	5	0.0185	4	0.0153
	$q=0.2$	12	0.0387	9	0.0320	10	0.0388	8	0.0252
	$q=0.5$	12	0.0386	9	0.0369	41	0.1256	34	0.1045
	$q=1$	12	0.0350	9	0.0310	151	0.3943	125	0.2940
$n=80$	$q=0.05$	11	0.2328	8	0.1636	3	0.0498	3	0.0416
	$q=0.1$	12	0.1973	9	0.1914	5	0.0799	4	0.0781
	$q=0.2$	12	0.1826	9	0.1793	10	0.1839	8	0.1569
	$q=0.5$	12	0.1752	9	0.1713	45	0.6057	38	0.5479
	$q=1$	13	0.2098	9	0.1809	161	2.1321	143	1.8236
$n=160$	$q=0.05$	11	1.0480	8	1.0091	3	0.2526	3	0.2465
	$q=0.1$	11	1.0756	9	1.0320	5	0.4143	3	0.3542
	$q=0.2$	12	1.1480	10	1.1212	9	0.9526	7	0.7437
	$q=0.5$	12	1.1716	9	1.1646	50	3.9229	40	3.2474
	$q=1$	13	1.3568	10	1.2345	158	12.5696	139	10.5460

and CPU time. As q becomes large (the skew-Hermitian part $\mathbf{S}$ is dominant), the superiorities of NHSS and NPHSS methods disappear.

In Table 5, numerical results for the inexact variants of the HSS, PHSS, NHSS and NPHSS methods are listed. Here we adopt the corresponding iteration parameters in Table 1 for convenience and not the experimental optimal parameters of its own. From Table 5, we can obtain the same conclusions as above Tables.

Table 4
Numerical results for HSS, PHSS, NHSS and NPHSS with the theoretical quasi-optimal iteration parameters.

Method	n	HSS		PHSS		NHSS		NPHSS	
q		IT	CPU	IT	CPU	IT	CPU	IT	CPU
$n = 10$	$q = 0.05$	11	0.0048	10	0.0043	3	0.0026	3	0.0014
	$q = 0.1$	11	0.0052	10	0.0052	5	0.0031	4	0.0017
	$q = 0.2$	10	0.0048	9	0.0045	9	0.0055	7	0.0060
	$q = 0.5$	9	0.0095	8	0.0067	31	0.0257	22	0.0146
	$q = 1$	8	0.0037	8	0.0031	95	0.0451	66	0.0283
$n = 20$	$q = 0.05$	13	0.0109	12	0.0106	4	0.0039	3	0.0026
	$q = 0.1$	13	0.0122	11	0.0108	7	0.0057	5	0.0044
	$q = 0.2$	13	0.0115	11	0.0110	17	0.0138	10	0.0081
	$q = 0.5$	11	0.0169	9	0.0120	71	0.0568	41	0.0265
	$q = 1$	10	0.0091	9	0.0076	218	0.1389	128	0.0836
$n = 40$	$q = 0.05$	14	0.0503	13	0.0490	4	0.0153	3	0.0131
	$q = 0.1$	14	0.0483	13	0.0470	8	0.0245	6	0.0171
	$q = 0.2$	14	0.0430	13	0.0397	23	0.0590	13	0.0429
	$q = 0.5$	13	0.0429	13	0.0395	116	0.3113	60	0.1698
	$q = 1$	12	0.0413	10	0.0354	353	0.8687	192	0.4649
$n = 80$	$q = 0.05$	15	0.2184	13	0.2123	4	0.0562	3	0.0450
	$q = 0.1$	15	0.2517	13	0.2054	9	0.1459	6	0.0809
	$q = 0.2$	15	0.2256	13	0.2104	25	0.3396	14	0.1883
	$q = 0.5$	14	0.2155	12	0.2099	134	1.8272	69	0.9498
	$q = 1$	14	0.1994	12	0.1951	495	6.4603	251	3.2606
$n = 160$	$q = 0.05$	15	1.4244	12	1.2974	4	0.3242	3	0.2355
	$q = 0.1$	15	1.4711	13	1.3944	9	0.7425	6	0.5190
	$q = 0.2$	15	1.5617	13	1.4626	26	2.0862	15	1.2312
	$q = 0.5$	15	1.4228	13	1.3990	140	10.8828	72	5.6789
	$q = 1$	14	1.3140	12	1.2006	539	42.3886	272	21.8806

Table 5
Numerical results for IHSS, IPHSS, INHSS and INPHSS.

Method	n	IHSS		IPHSS		INHSS		INPHSS	
q	CPU	IT	CPU	IT	CPU	IT	CPU	IT	CPU
$n = 10$	$q = 0.05$	9	0.0031	7	0.0026	3	0.0009	3	0.0008
	$q = 0.1$	9	0.0037	6	0.0032	4	0.0013	3	0.0009
	$q = 0.2$	9	0.0040	7	0.0031	6	0.0029	5	0.0023
	$q = 0.5$	8	0.0030	6	0.0025	19	0.0102	17	0.0065
	$q = 1$	8	0.0041	7	0.0033	60	0.0192	50	0.0135
$n = 20$	$q = 0.05$	11	0.0080	9	0.0068	3	0.0031	3	0.0022
	$q = 0.1$	10	0.0124	8	0.0101	5	0.0037	4	0.0030
	$q = 0.2$	11	0.0083	9	0.0071	9	0.0070	8	0.0062
	$q = 0.5$	10	0.0090	8	0.0079	33	0.0297	29	0.0265
	$q = 1$	9	0.0134	7	0.0115	109	0.0721	98	0.0634
$n = 40$	$q = 0.05$	11	0.0561	8	0.0234	3	0.0115	3	0.0108
	$q = 0.1$	12	0.0342	9	0.0259	5	0.0158	4	0.0135
	$q = 0.2$	12	0.0301	9	0.0276	9	0.0289	8	0.0240
	$q = 0.5$	12	0.0351	9	0.0330	35	0.1003	30	0.0927
	$q = 1$	12	0.0313	9	0.0278	134	0.3210	110	0.2786
$n = 80$	$q = 0.05$	11	0.2012	8	0.1394	3	0.0379	3	0.0359
	$q = 0.1$	12	0.1620	9	0.1413	5	0.0615	4	0.0585
	$q = 0.2$	11	0.1541	9	0.1303	9	0.1419	7	0.1229
	$q = 0.5$	11	0.1483	9	0.1276	37	0.5643	31	0.5021
	$q = 1$	12	0.1756	9	0.1629	141	1.7351	129	1.5330
$n = 160$	$q = 0.05$	11	0.9487	8	0.9094	3	0.2223	3	0.2198
	$q = 0.1$	11	0.9786	9	0.9110	5	0.3753	3	0.3257
	$q = 0.2$	12	1.0100	10	0.9846	9	0.8528	6	0.6215
	$q = 0.5$	11	1.0158	9	0.9761	39	3.2992	34	2.9990
	$q = 1$	12	1.1586	10	1.0001	139	10.3599	128	8.9660

Therefore, the PHSS and NPHSS methods proposed in this work are two powerful and attractive iterative approaches for solving large sparse continuous Sylvester equations. In addition, we tend to use NPHSS iteration method to solve continuous Sylvester equation (1) when the Hermitian parts of the two coefficient matrices are dominant and employ PHSS if the skew-Hermitian parts are dominant.

5. Conclusions

For solving a broad class of continuous Sylvester equations, we have proposed a preconditioned HSS (PHSS) iteration method and its non-alternating variant (NPHSS) to improve the convergence efficiency of the HSS iteration method [27]. The PHSS method is obviously a type of generalization of the classical HSS method [27]. PHSS returns to HSS when we take $P_1 = I_m, P_2 = \frac{\beta}{\alpha} I_n$. The NPHSS method is an efficient non-alternating variant of the PHSS method. Moreover, we establish the inexact variants of the PHSS and NPHSS iteration methods to reduce the computational cost. Convergence properties of the PHSS and NPHSS iteration methods and their inexact variants are analyzed in detail. We also give the theoretical quasi-optimal parameters that minimize the upper bounds of the iterative spectrums of the PHSS and NPHSS methods. Finally, we give some numerical results to illustrate the efficiency and robustness of our methods.

At last, we should mention that the choice of the optimal iteration parameters for the PHSS and NPHSS methods is still an interesting but difficult topic, which may be considered in future study.

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