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Dynamics for an SIRS epidemic model with infection age and relapse on a scale-free network☆

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Abstract

A new SIRS epidemic model with infection age and relapse on a scale-free network is introduced. The basic reproduction number R_0 is defined. Asymptotic smoothness of solution and uniform persistence of system are proved. It is shown that the disease-free equilibrium is globally asymptotically stable by using Fluctuation Lemma if $R_0 < 1$ and the endemic equilibrium is globally asymptotically stable by constructing suitable Lyapunov functional if $R_0 > 1$. Effects of two immunization schemes are studied. Numerical simulations and sensitivity analysis are performed. Results manifest that infection age and degree of node play a significant role in controlling the emergence and spread of the epidemic disease.

1. Introduction

As revealed by the transmission history of epidemic disease, epidemic disease is increasingly threatening health of people and destroying harmony of society. Therefore, it has aroused extensive attentions of many researchers. In order to enact appropriate control strategies, compartmental models are proposed. Kermack and McKendrick [1–3] firstly came up with classical SIS and SIR compartment models. Most of these models are computed under the assumption that everyone within a compartment behaves identically, regardless of how much

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time they have spent in their compartment. For example, it suppose that everyone in infected compartment has the same waiting time. In fact, the waiting time for everyone in infected compartment differs from one to one, which may rely on the types of epidemic disease and individuals' status. To be more realistic and relevant, some compartment models are generalized with continuous age assumption by introducing infection age.

Recently, many works [4–15] have been made for investigating the transmission dynamics of epidemic disease with age structured. Li et al. [5] considered an SIVS epidemic model with treatment and age of vaccination, and they obtained the stability and bifurcation of the model. McCluskey [6] studied a model of disease transmission with continuous age structured for latently infected individuals and infectious individuals and obtained that disease-free equilibrium and endemic equilibrium were globally asymptotically stable. Chen et al. [7] considered an SIRS model with infection age and proved that disease-free and endemic equilibria were globally asymptotically stable. Duan et al. [8] considered an SVIR epidemic model with age of vaccination and shown that the global stability of the infection-free and the endemic equilibria depended only on the basic reproductive number $R_0(\psi)$. Liu et al. [9] proposed an SEIR epidemic model for a disease with age dependent latency and relapse and obtained the local stability and global stability of the infection-free and the endemic equilibria. Chen et al. [10] studied an SIR epidemic model with infection age and saturated incidence and proved that disease-free and endemic equilibria were globally asymptotically stable. Chu et al. [11] applied the recently developed normal form theory for abstract Cauchy problem with non-dense domain [12] to study normal form for an age structured model and provided detailed computations for the Taylor's expansion of the reduced system on the center manifold. Thieme [13] introduced concepts of "integrated semigroups" which have been developed in the theory of weakly continuous semigroups on dual Banach spaces, and researched the relation to integrated solutions of homogeneous and inhomogeneous abstract Cauchy problems. Magal et al. [14] analyzed the global asymptotic behavior of a two-group SI (susceptible and infected) epidemic model with age of infection. They proved that the model exhibited the traditional threshold behavior where the disease-free equilibrium was globally asymptotically stable if the basic reproduction number was less than one, and the endemic equilibrium was globally asymptotically stable if the basic reproduction number was greater than one. Magal [15] investigated the existence of compact attractors for time-periodic age-structured models and researched the eventual compactness of a class of abstract non-autonomous semiflow.

However, above models have an important assumption that everyone is uniformly mixed and homogenous contacts. Since the contact process of population can not be always uniform collision, epidemic disease transmission is usually heterogeneous. Therefore, the idea of complex network is introduced to epidemic model. In recent years, many attempts [16–20] have been made for investigating the transmission dynamics of epidemic disease on complex network. Li and Shuai [16] investigated the global-stability problem of equilibria for the coupled system of differential equations on network and the systematic approach was applied to several classes of coupled systems in engineering, ecology and epidemiology. Liu and Zhang [17] presented an SEIRS epidemic model on the scale-free network and proved the local stability of disease-free equilibrium and the permanence of the disease on the network. Zhang and Jin [18] considered an epidemic model with birth and death on network and the stabilities of the equilibria were analysed. Jin et al. [19] proposed and studied network epidemic model with demographics for disease transmission and found that demographics had a great effect on basic reproduction number R_0. Huang et al. [20] investigated a new SIQRS epidemic model with demographics and vaccination on complex heterogeneous network and proved the permanence of the disease. Zhang et al. [21] studied a seasonal influenza-like disease model by incorporating the interplay between subsidy policies and human behavioral responses, and their findings showed that the targeted subsidy policy was only advantageous when individuals prefered to imitate the subsidized individuals strategy. Zhang and Fu [22] studied the spreading of epidemics on a scale-free networks with infectivity which was nonlinear in the connectivity of nodes, and showed that the nonlinear infectivity was more appropriate than constant or linear ones, and gived the epidemic threshold of the SIS model on a scale-free network with nonlinear infectivity. Yang et al. [23] proposed an SIS epidemic model with infection age on complex network and proved stability of steady state. Yang and Chen [24] mainly studied the SIR epidemic model with infection age on the complex network and proved stability of steady state, but they ignored that the removed individuals were temporary immunity and relapse. Other epidemic models or drinking models with or without complex networks, please see [25–31] and references cited therein.

Motivated by the above discussions, the goal of the present paper is to construct a more realistic SIRS epidemic model with the infection age and relapse on the scale-free network, in which we assume that removed individuals are temporary immunity and relapse. We obtain that the globally asymptotical stability of the disease-free equilibrium if $R_0 < 1$ by using Fluctuation Lemma and the globally asymptotical stability of the endemic equilibrium if $R_0 > 1$ by constructing suitable Lyapunov functional. Our results indicate that infection age and degree of node play a significant role in controlling the emergence and spread of the epidemic disease.

The remain part of this paper is organized in the following: In Section 2, we introduce a new *SIRS* epidemic model with infection age and relapse on the scale-free network. In Section 3, we define the basic reproductive number, and prove the globally asymptotical stabilities of disease-free and endemic equilibrium. In Section 4, we investigate and compare two major immunization strategies, uniform immunization and targeted immunization. In Section 5, we carry out some numerical simulations. In the last section, we perform sensitivity analysis on a few parameters and make some discussions.

2. Mathematical model

2.1. System description

The total population N is divided into n groups according to the degree of node, namely,

$$N = N_1 + N_2 + \cdots + N_n. \tag{2.1}$$

Furthermore, we divide everyone groups into three compartments:$S_k(t)$, $I_k(t, a)$, $R_k(t)$. $S_k(t)$ represents the density of susceptible vertices of degree k at time t; Let $I_k(t, 0)$ denote infected vertices of degree k who have just become infected at time t. a is age of infection. As time progresses, this group of people has the same age of infection a, so $I_k(t, a)$ denotes the density of infected vertices of degree k with age of infection a at time t. If we consider time frames that are comparable with the age of infection of all the population, we can assume that the age is infinity [32,33]. Then, the total population of infected vertices of degree k in all infection-age classes is $\int_0^\infty I_k(t,a)da$; $R_k(t)$ represents the density of removed vertices of degree k at time t. $N_k(t)$ represents the total number of vertices of degree k at time t and

$$N_k(t) = S_k(t) + \int_0^\infty I_k(t,a)da + R_k(t), \tag{2.2}$$

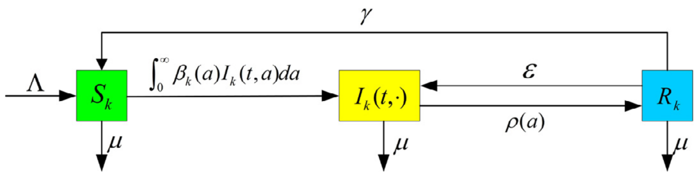

Fig. 1. Flowchart for an *SIRS* epidemic model with infection age and relapse on a scale-free network.

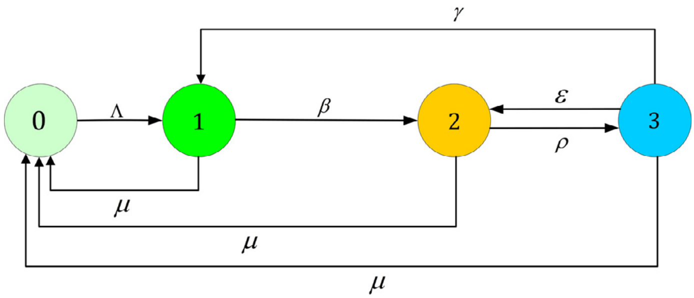

Fig. 2. Flowchart for an *SIRS* epidemic model with infection age and relapse and supposing that there is an empty node.

where n is the maximal degree of the scale-free network and $k \in \mathbb{N}_n \triangleq \{1, 2, \ldots, n\}$.

The model structure is shown in Fig. 1.

According to [24,34–36], we suppose that every individual is spatially distributed on the network. Each node of the network is empty or occupied by at most one individual. We give each node a number: 0, 1, 2 or 3. 0 means empty state; 1 is occupied a susceptible individual; 2 is occupied a infected individual and 3 is occupied a removed individual. The state of the system at time t can be described by a set of numbers, 0, 1, 2 or 3. Each node changes its state with a certain rate. An empty node gives birth to a susceptible individual at rate Λ. A susceptible individual can be infected by contact at rate β if there are infected individuals in its nearest neighbors. An infected individual can be cured at rate ρ. A removed individual can be relapsed at rate ε. A removed individual can be transformed susceptible individual at rate γ. μ is the natural death rate. If an individual dies, there is an empty node left. The model structure is shown in Fig. 2.

The transfer diagram leads to the following system of ordinary and partial differential equations:

$$
\begin{cases}
\frac{dS_k(t)}{dt} = \Lambda k(1 - N_k(t))\varphi(t) + \gamma R_k(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t), & (2.3a)\\
\frac{\partial I_k(t,a)}{\partial t} + \frac{\partial I_k(t,a)}{\partial a} = -\rho(a)I_k(t,a) - \mu I_k(t,a), & (2.3b)\\
\frac{dR_k(t)}{dt} = \int_0^\infty \rho(a)I_k(t,a)da - (\gamma + \mu + \varepsilon)R_k(t), & (2.3c)
\end{cases}
$$

with the boundary condition $I_k(t, 0) = k\sigma S_k(t)\Theta(I_k(t,\cdot)) + \varepsilon R_k(t)$ for $t \geq 0$ and the initial conditions $S_k(0) = S_{k0}, I_k(0, a) = I_{k0}(a), R_k(0) = R_{k0}$ for $a \geq 0$ and $k \in \mathbb{N}_n$. Here, $S_{k0} \in (0, \infty)$, $R_{k0} \in [0, \infty)$ and $I_{k0}(a) \in L^1_+[0, \infty)$, where $L^1_+[0, \infty)$ is the space of function on $[0, \infty)$ that are nonnegative and Lebesgue integrable.

σ is the effective exposure rate of a susceptible to the infected individuals; $\beta_k(a)$ is the transmission rate vertices of degree k with infection age a from the susceptible people to the infected people; $\rho(a)$ is the transmission rate with infection age a from the infected people to the removed people. $\Theta(I_k(t,\ \cdot))$ which means the infection force is denoted by

$$\Theta(I_k(t,\cdot)) = \frac{\sum_{k=1}^{n}(k-1)p(k)\int_0^\infty \beta_k(a)I_k(t,a)da}{\langle k\rangle},\ \ k\in\mathbb{N}_n. \tag{2.4}$$

Similar to be [19], the probability of fertility contacts between nodes with degree k and their neighbours with degree j is

$$\varphi(t) = \sum_{j=1}^{n} p(j|k)\frac{1}{j}N_j(t),\ \ k, j\in\mathbb{N}_n. \tag{2.5}$$

$\Lambda k(1-N_k(t))\varphi(t)$ represents the density of new born individuals per unit time for $k\in\mathbb{N}_n$. For the degree uncorrelated scale-free network, the conditional probability of a node with degree k linking to a node with degree j is $p(j|k)=\frac{jp(j)}{\langle k\rangle}$, where the degree distribution $p(k)=cf(k)k^{-r}$, $(2<r\le 3)$, c is any constant satisfying $\sum_{k=1}^{n}p(k)=1$, and the value $\langle k\rangle=\sum_{k=1}^{n}kp(k)$ is the mean degree. Due to each infected node, there must be one edge pointing to another infected node, the conditional probability of an infected node with degree j linking freely with a susceptible with degree k is $p'(j|k)=\frac{(j-1)p(j)}{\langle k\rangle}$, $k, j\in\mathbb{N}_n$.

From system (2.3), we obtain

$$\begin{aligned}
\frac{dN_k(t)}{dt} =& \frac{dS_k(t)}{dt} + \int_0^\infty \frac{\partial I_k(t,a)}{\partial t}da + \frac{dR_k(t)}{dt}\\
=&\ \Lambda k(1-N_k(t))\varphi(t) + \gamma R_k(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t) + \int_0^\infty \frac{\partial I_k(t,a)}{\partial t}da\\
&+ \int_0^\infty \rho(a)I_k(t,a)da - (\gamma+\mu+\varepsilon)R_k(t)\\
=&\ \Lambda k(1-N_k(t))\varphi(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t) + \int_0^\infty \rho(a)I_k(t,a)da - (\mu+\varepsilon)R_k(t)\\
&- \int_0^\infty \left(\rho(a)I_k(t,a) + \mu I_k(t,a) + \frac{\partial I_k(t,a)}{\partial a}\right)da\\
=&\ \Lambda k(1-N_k(t))\varphi(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t) - (\mu+\varepsilon)R_k(t)\\
&- \mu\int_0^\infty I_k(t,a)da - \int_0^\infty \frac{\partial I_k(t,a)}{\partial a}da\\
=&\ \Lambda k(1-N_k(t))\varphi(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t) - (\mu+\varepsilon)R_k(t)\\
&- \mu\int_0^\infty I_k(t,a)da + I_k(t,0)\\
=&\ \Lambda k(1-N_k(t))\varphi(t) - \mu S_k(t) - \mu R_k(t) - \mu\int_0^\infty I_k(t,a)da\\
=&\ \Lambda k\varphi(t) - (\Lambda k\varphi(t)+\mu)N_k(t),
\end{aligned} \tag{2.6}$$

for $k\in\mathbb{N}_n$. According to [24,37], when $\Lambda>\mu$, Eq. (2.6) has a positive equilibrium meeting $N_k^*=\frac{\Lambda k\varphi^*}{\mu+\Lambda k\varphi^*}$, $k\in\mathbb{N}_n$. Pluging N_k^* into φ^*, we yeild

$$h(\varphi) = \frac{1}{\langle k\rangle}\sum_{k=1}^{n}\frac{kp(k)\Lambda\varphi}{\mu+\Lambda k\varphi},\ \ k\in\mathbb{N}_n. \tag{2.7}$$

Note that

$$h(1) = \frac{1}{\langle k\rangle}\sum_{k=1}^{n}\frac{kp(k)\Lambda}{\mu+\Lambda k} < 1,\ \ k\in\mathbb{N}_n, \tag{2.8}$$

$$h'(\varphi) = \frac{1}{\langle k\rangle}\sum_{k=1}^{n}\frac{kp(k)\Lambda\mu}{(\mu+\Lambda k\varphi)^2} > 0,\ \ k\in\mathbb{N}_n, \tag{2.9}$$

thus $h'(0)=\frac{\Lambda}{\mu}>1$ if $\Lambda>\mu$. This indicates that h has a unique fixed point φ^*. Therefore,

$$\lim_{t\to\infty}N_k(t) = \frac{\Lambda k\varphi^*}{\mu+\Lambda k\varphi^*},\ \ k\in\mathbb{N}_n. \tag{2.10}$$

If $\Lambda\le\mu$, there is $\lim_{t\to\infty}N_k(t)=0, k\in\mathbb{N}_n$. Furthermore, the population becomes extinct and there is no other dynamical behaviours any more. Therefore, we only need to consider the following limiting system which satisfies $S_k(t)+\int_0^\infty I_k(t,a)da+R_k(t)=N_k^*$ as following,

$$\begin{cases}\frac{dS_k(t)}{dt} = \Lambda k\big(1-N_k^*\big)\varphi^* + \gamma R_k(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t), & (2.11a)\\ \frac{\partial I_k(t,a)}{\partial t} + \frac{\partial I_k(t,a)}{\partial a} = -\rho(a)I_k(t,a) - \mu I_k(t,a), & (2.11b)\\ \frac{dR_k(t)}{dt} = \int_0^\infty \rho(a)I_k(t,a)da - (\gamma+\mu+\varepsilon)R_k(t), & (2.11c)\end{cases}$$

with the boundary condition $I_k(t,0)=k\sigma S_k(t)\Theta(I_k(t,\cdot))+\varepsilon R_k(t)$ for $t\ge0$ and the initial conditions $S_k(0)=S_{k0}, I_k(0,a)=I_{k0}(a), R_k(0)=R_{k0}$ for $a\ge0$ and $k\in\mathbb{N}_n$. Here, $S_{k0}\in(0,\infty)$, $R_{k0}\in[0,\infty)$ and $I_{k0}(a)\in L_+^1[0,\infty)$, where $L_+^1[0,\infty)$ is the space of function on $[0,\infty)$ that are nonnegative and Lebesgue integrable.

Assumption 2.1.1. For system (2.11), we assume that

(i) $\rho(a)\in L_+^\infty[0,\infty)$ with essential upper bound $\overline{\rho}$ and $\beta_k(a)\in L_+^\infty[0,\infty)\bigcap U_{BC}([0,\infty),[0,\infty))$ $(k\in\mathbb{N}_n)$, where $U_{BC}([0,\infty),[0,\infty))$ is a set of all bounded and uniformly continuous functions from $[0,\infty)$ to $[0,\infty)$ with essential upper bounds $\overline{\beta_k}$, namely,

$$\overline{\rho} = \operatorname*{ess\,sup}_{a\in[0,\infty)}\rho(a) < \infty, \overline{\beta_k} = \operatorname*{ess\,sup}_{a\in[0,\infty)}\beta_k(a) < \infty;$$

(ii) $\rho(a)$, $\beta_k(a)$ are Lipschitz continuous on $[0,\infty)$ and $\beta_k(a)$ is nondecreasing, $k\in\mathbb{N}_n$;

(iii) there exist $m_0\in(0,m]$ and $m_{k0}\in(0,m_k]$, such that $\rho(a)\ge m_0$ and $\beta_k(a)\ge m_{k0}$ for $a\ge0$, $k\in\mathbb{N}_n$;

(iv) $\lim_{t\to\infty}\left\|\frac{\pi(t+\cdot)}{\pi(\cdot)}\right\|_\infty=0$, where $\pi(a)=e^{-\int_0^a\phi(s)ds}$, $\phi(a)=\rho(a)+\mu$, for $a\in[0,\infty)$, is the probability of the infected individuals still staying in the infected compartment;

(v) system (2.11) satisfies the coupling condition $I_{k0}(0)=k\sigma S_{k0}\Theta(I_{k0})$, for $k\in\mathbb{N}_n$.

2.2. Basic properties

To show that system (2.11) is epidemiologically meaningful, we will prove that all solutions of system (2.11) with initial conditions $S_{k0}>0$, $0\le\int_0^\infty I_{k0}(a)da<\infty$, and $R_{k0}\ge0$ are nonnegative and bounded for $t>0$ and $k\in\mathbb{N}_n$. Thus, we have the following Lemmas.

2.2.1. Positivity of solutions and infection force

Lemma 2.2.1. *Solutions $S_k(t)$ and $R_k(t)$ of system (2.11) with initial conditions $S_{k0}>0$ and $R_{k0}\geq 0$ are positive for $t>0$ and $k\in\mathbb{N}_n$.*

Proof. Similar to Appendix A of [20], we will prove that $S_k(t)>0$ for $t>0$ and $k\in\mathbb{N}_n$. Assume that $S_k(t)$ is not always positive when $t>0$ and $k\in\mathbb{N}_n$. Notice that $S_k(0)>0$. Due to Eq. (4.2a) and the continuity of $S_k(t)$, there is a sufficiently small $\varepsilon>0$, such that $S_k(t)>0$ for $t\in(0,\ \varepsilon)$ and $k\in\mathbb{N}_n$. Furthermore, there exist $j\in\mathbb{N}_n$ and the first time $t_1\geq\varepsilon>0$, such that $S_j(t_1)=0$ and $S_j(t)>0$ for $t\in(0,\ t_1)$, $k\in\mathbb{N}_n$. Combining with Eq. (4.2a), we reach

$$S_j'(t)+(\mu+k\sigma\Theta(I_j(t,\cdot)))S_j(t)>0,\ \ t\in(0,t_1),\ \ k\in\mathbb{N}_n. \tag{2.12}$$

Therefore, we have

$$S_j(t)>S_j(0)e^{-(\mu+k\sigma)t}\geq 0,\ \ t\in(0,t_1),\ \ k\in\mathbb{N}_n. \tag{2.13}$$

From Eq. (4.2c), we obtain

$$R_j'(t)+(\gamma+\mu)R_j(t)>0,\ \ t\in(0,t_1),\ \ k\in\mathbb{N}_n. \tag{2.14}$$

Thus, we have

$$R_j(t)>R_j(0)e^{-(\gamma+\mu)t}\geq 0,\ \ t\in(0,t_1),\ \ k\in\mathbb{N}_n. \tag{2.15}$$

Obviously, due to the continuity of $R_j(t)$, $j\in\mathbb{N}_n$, we yield $R_j(t_1)\geq 0$, $j\in\mathbb{N}_n$. Furthermore, Eq. (4.2a) indicates that

$$S_j'(t_1)=\Lambda j\big(1-N_j^*\big)\varphi^*+\gamma R_j(t_1)>0,\ \ j\in\mathbb{N}_n. \tag{2.16}$$

Thus,

$$S_j(t)<S_j(t_1)=0,\ \ t\in(t_1-\tau,t_1)\subset(0,t_1),\ \ j\in\mathbb{N}_n, \tag{2.17}$$

where τ is an arbitrarily positive constant. It is an apparent contradiction. By using the method of step-to-step, we have $S_k(t)>0$ for $t>0$ and $k\in\mathbb{N}_n$.

Correspondingly, from Eq. (4.2c), we get $R_k(t)>0$ for $t>0$ and $k\in\mathbb{N}_n$. This completes the proof. □

Lemma 2.2.2. *$\Theta(I_k(t,\ \cdot))$ with $\Theta(I_{k0})>0$ and solution $\int_0^\infty I_k(t,a)da$ of system (2.11) with initial condition $0\leq\int_0^\infty I_{k0}(a)da<\infty$ are positive for $t>0$ and $k\in\mathbb{N}_n$.*

Proof. According to the definition of $\Theta(I_k(t,\ \cdot))$, we have

$$\begin{aligned}
\frac{d\Theta(I_k(t,\cdot))}{dt}&=\langle k\rangle^{-1}\left(\sum_{k=1}^n(k-1)p(k)\int_0^\infty\beta_k(a)\frac{\partial I_k(t,a)}{\partial t}da\right)\\
&=\langle k\rangle^{-1}\left(\sum_{k=1}^n(k-1)p(k)\int_0^\infty\beta_k(a)\left(-\rho(a)I_k(t,a)-\mu I_k(t,a)-\frac{\partial I_k(t,a)}{\partial a}\right)da\right)\\
&\geq-\overline{\rho}\Theta(I_k(t,\cdot))-\mu\Theta(I_k(t,\cdot))-\langle k\rangle^{-1}\left(\sum_{k=1}^n(k-1)p(k)\int_0^\infty\beta_k(a)\frac{\partial I_k(t,a)}{\partial a}da\right)\\
&=-(\overline{\rho}+\mu)\Theta(I_k(t,\cdot))\\
&\quad-\langle k\rangle^{-1}\left(\sum_{k=1}^n(k-1)p(k)\left(\beta_k(a)I_k(t,a)|_{a=0}^{a=\infty}-\int_0^\infty I_k(t,a)d\beta_k(a)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq -(\overline{\rho}+\mu)\Theta(I_k(t,\cdot)) + \langle k\rangle^{-1}\left(\sum_{k=1}^{n}(k-1)p(k)\beta_k(0)I_k(t,0)\right)\\
&= -(\overline{\rho}+\mu)\Theta(I_k(t,\cdot))\\
&\quad + \langle k\rangle^{-1}\left(\sum_{k=1}^{n}(k-1)p(k)\beta_k(0)(k\sigma S_k(t)\Theta(I_k(t,\cdot)) + \varepsilon R_k(t))\right),
\end{aligned}
$$

for $k \in \mathbb{N}_n$. Clearly $\Theta(I_{k0})>0$, $S_k(t)>0$, $R_k(t)>0$, for $t>0, k \in \mathbb{N}_n$. This, combining with the above differential inequality, we get $\Theta(I_k(t,\cdot)) > \Theta(I_{k0})e^{-(\overline{\rho}+\mu)t} > 0$, for $t>0$ and $k \in \mathbb{N}_n$.

Next, for $k \in \mathbb{N}_n$, it follows from system (2.11) that

$$
\begin{aligned}
\frac{d\int_0^\infty I_k(t,a)da}{dt} &= \int_0^\infty \frac{\partial I_k(t,a)}{\partial t}da\\
&= -\int_0^\infty \left(\rho(a)I_k(t,a) + \mu I_k(t,a) + \frac{\partial I_k(t,a)}{\partial a}\right)da\\
&\geq -(\overline{\rho}+\mu)\int_0^\infty I_k(t,a)da - \int_0^\infty \frac{\partial I_k(t,a)}{\partial a}da\\
&= -(\overline{\rho}+\mu)\int_0^\infty I_k(t,a)da + I_k(t,0)\\
&= -(\overline{\rho}+\mu)\int_0^\infty I_k(t,a)da + k\sigma S_k(t)\Theta(I_k(t,\cdot)) + \varepsilon R_k(t),
\end{aligned}
$$

for $k \in \mathbb{N}_n$. As $\Theta(I_k(t,\ \cdot))>0$, $S_k(t)>0$, $R_k(t)>0$, for $t\in(0,\ \infty)$ and $k \in \mathbb{N}_n$, it follows that $\int_0^\infty I_k(t,a)da > 0$, for $t>0$ and $k \in \mathbb{N}_n$. This completes the proof. □

2.2.2. Invariant region

Define the space of function X as

$$X = \left(\mathbb{R}^+\right)^n \times \left(L^1_+[0,\infty)\right)^n \times \left(\mathbb{R}^+\right)^n, \tag{2.18}$$

which is equipped with the norm

$$\|(x_1,x_2,x_3)\| = \sum_{k=1}^{n}\left(|x_1| + \int_0^\infty |x_2(a)|da + |x_3|\right). \tag{2.19}$$

The initial conditions $S_{k0}>0$, $0 \leq \int_0^\infty I_{k0}(a)da < \infty$, $R_{k0}\geq 0$ for $k \in \mathbb{N}_n$ that belong to the positive cone of X can be rewritten as $x_{k0} = (S_{k0}, I_{k0}(\cdot), R_{k0}) \in X$ for $k \in \mathbb{N}_n$. According to the theory of functional differential equation [38], it is clearly proved that system (2.11) with initial conditions $S_{k0}>0$, $0 \leq \int_0^\infty I_{k0}(a)da < \infty$, $R_{k0}\geq 0$ for $k \in \mathbb{N}_n$ has a unique nonnegative solution. Therefore, we have a continuous semi-flow associated with system (2.11), namely, $\Phi : \left(\mathbb{R}^+\right)^n \times X \to X$ which is generated by system (2.11) takes the following form

$$\Phi(t, x_{k0}) = \Phi_t(x_{k0}) = (S_k(t), I_k(t,\cdot), R_k(t))\ t \geq 0, x_{k0} \in X,\ \ k \in \mathbb{N}_n, \tag{2.20}$$

with

$$\|\Phi_t(x_{k0})\|_X = \|S_k(t), I_k(t,\cdot), R_k(t)\|_X = \sum_{k=1}^{n}\left(|S_k(t)| + \int_0^\infty |I_k(t,a)|da + |R_k(t)|\right), \tag{2.21}$$

for $a\geq 0$ and $k \in \mathbb{N}_n$.

Set

$$\Gamma = \left\{ \left(S_k(t), \int_0^{\infty} I_k(t,a)da, R_k(t) \right) \in X : 0 \leq S_k(t), \int_0^{\infty} I_k(t,a)da, R_k(t) \right.$$
$$\left. \leq S_k(t) + \int_0^{\infty} I_k(t,a)da + R_k(t) \leq 1 \right\},$$

for $k \in \mathbb{N}_n$. Thus, Γ is a positive invariance for Φ, i.e., $\Phi(t,\ x_{k0}) \in \Gamma$, $\forall t \geq 0$, $x_{k0} \in \Gamma$, for $k \in \mathbb{N}_n$ and Γ is point dissipative and attracts all solutions with initial conditions $S_{k0} > 0$, $0 \leq \int_0^{\infty} I_{k0}(a)da < \infty$, $R_{k0} \geq 0$ of system (2.11) in X, for $k \in \mathbb{N}_n$.

3. Analysis of the model

3.1. Disease-free equilibrium and the basic reproductive number

Clearly, the disease-free equilibrium of system (2.11) is

$$E^0 = \left\{ \left(S_k^0, I_k^0(\cdot), R_k^0 \right) \right\}_k = \left\{ \left(\frac{\Lambda k(1 - N_k^*)\varphi^*}{\mu}, 0, 0 \right) \right\}_k, k \in \mathbb{N}_n. \tag{3.1}$$

Next, according to the biological meaning of R_0, we define the form of R_0 as following,

$$R_0 = \frac{\sigma \sum_{j=1}^{n} j(j-1) S_j^0 p(j) \int_0^{\infty} \beta_j(a)\pi(a)da}{\langle k \rangle}, k \in \mathbb{N}_n, \tag{3.2}$$

where $\sum_{j=1}^{n} j(j-1)/\langle k \rangle$ denotes the total contacts by an infected individual in the network consisting of susceptible individuals only. $\beta_j(a)\pi(a)$ is the transmission ability of an infected individual still staying in the infected compartment with infection age a. Hence, $\int_0^{\infty} \beta_j(a)\pi(a)da$ denotes the total transmission ability of an infected individual during the infected period and $\sigma \int_0^{\infty} \beta_j(a)\pi(a)da$ means the effective transmission ability.

3.2. Stability of disease-free equilibrium

Theorem 3.2.1. *Disease-free equilibrium E^0 of system (2.11) is locally asymptotically stable if $R_0 < 1$, and is unstable if $R_0 > 1$.*

Proof. The linearized system of system (2.11) around the disease-free equilibrium $E^0 = \left\{ \left(S_k^0, I_k^0(\cdot), R_k^0 \right) \right\}_k$, $k \in \mathbb{N}_n$ is

$$\begin{cases} \frac{dS_k(t)}{dt} = -\mu S_k(t) - k\sigma S_k^0 \Theta(I_k(t,\cdot)) + \gamma R_k(t), & (3.3a) \\ \frac{\partial I_k(t,a)}{\partial t} + \frac{\partial I_k(t,a)}{\partial a} = -\rho(a) I_k(t,a) - \mu I_k(t,a), & (3.3b) \\ \frac{dR_k(t)}{dt} = \int_0^{\infty} \rho(a) I_k(t,a)da - (\gamma + \mu + \varepsilon) R_k(t), & (3.3c) \end{cases}$$

with the boundary condition $I_k(t,0) = k\sigma S_k^0 \Theta(I_k(t,\cdot)) + \varepsilon R_k(t)$ for $k \in \mathbb{N}_n$. Substituting $S_k(t) = S_{k0} e^{\lambda t}$, $I_k(t,a) = I_k(a) e^{\lambda t}$, $R_k(t) = R_{k0} e^{\lambda t}$, $k \in \mathbb{N}_n$ into system (3.3), we have

$$\begin{cases} 0 = -(\lambda + \mu) S_{k0} - k\sigma S_k^0 \Theta(I_k(\cdot)) + \gamma R_{k0}, & (3.4a) \\ \frac{dI_k(a)}{da} = -(\lambda + \rho(a) + \mu) I_k(a), & (3.4b) \\ 0 = \int_0^{\infty} \rho(a) I_k(a)da - (\lambda + \gamma + \mu + \varepsilon) R_{k0}, & (3.4c) \end{cases}$$

with the boundary condition $I_k(0) = k\sigma S_k^0 \Theta(I_k(\cdot)) + \varepsilon R_k(t)$, for $k \in \mathbb{N}_n$. Therefore, the characteristic equation of system (2.11) is

$$\begin{vmatrix} (\lambda+\mu)E_n & * & -\gamma E_n \\ 0 & E_n - A & 0 \\ 0 & 0 & (\lambda+\gamma+\mu+\varepsilon)E_n \end{vmatrix} = 0, \tag{3.5}$$

for $k \in \mathbb{N}_n$, where $E_n - A =$

$$\begin{vmatrix} 1 & -\frac{S_1^0 p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & -\frac{2S_1^0 p(3)\int_0^\infty \beta_3(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \cdots & -\frac{(n-1)S_1^0 p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \\ 0 & 1-\frac{2S_2^0 p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & -\frac{4S_2^0 p(3)\int_0^\infty \beta_3(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \cdots & -\frac{2(n-1)S_2^0 p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\frac{nS_n^0 p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & -\frac{2nS_n^0 p(3)\int_0^\infty \beta_3(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \cdots & 1-\frac{n(n-1)S_n^0 p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \end{vmatrix}. \tag{3.6}$$

By simple calculation, we have

$$1 - \frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)e^{-\lambda a}\pi(a)da}{\langle k\rangle} = 0, \tag{3.7}$$

for $k, j \in \mathbb{N}_n$. We suppose that Eq. (3.7) has a root λ_0 with $Re(\lambda_0) \geq 0$ if $R_0 < 1$. Thus, we have

$$1 - \frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)e^{-\lambda_0 a}\pi(a)da}{\langle k\rangle} = 0, \tag{3.8}$$

for $k, j \in \mathbb{N}_n$. Then

$$1 = \left|\frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)e^{-\lambda_0 a}\pi(a)da}{\langle k\rangle}\right| \leq \frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)\pi(a)da}{\langle k\rangle}$$
$$= R_0, \tag{3.9}$$

for $k, j \in \mathbb{N}_n$. It is an apparent contradiction. Thus, we obtain that all roots of Eq. (3.7) have negative real parts if $R_0 < 1$. According to Routh–Hurwitz criteria [39], we prove that the disease-free E^0 is locally asymptotically stable if $R_0 < 1$.

Next, if $R_0 > 1$, we denote the left-hand side of Eq. (3.7) as $F(\lambda)$. Namely,

$$F(\lambda) \triangleq 1 - \frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)e^{-\lambda a}\pi(a)da}{\langle k\rangle}, \tag{3.10}$$

for $k, j \in \mathbb{N}_n$. Obviously, $F(\lambda)$ is a continuously integral function and satisfies

$$F(0^+) = 1 - \frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)e^{-(0^+ a)}\pi(a)da}{\langle k\rangle} = 1 - R_0 < 0, \tag{3.11}$$

and

$$F(+\infty) = 1 - \frac{\sum_{j=1}^n j(j-1)S_j^0 p(j)\int_0^\infty \beta_k(a)e^{-(+\infty a)}\pi(a)da}{\langle k\rangle} = 1, \tag{3.12}$$

for $k, j \in \mathbb{N}_n$. Therefore, Eq. (3.7) has at least one positive root, then the disease-free E^0 is unstable if $R_0 > 1$. This completes the proof. □

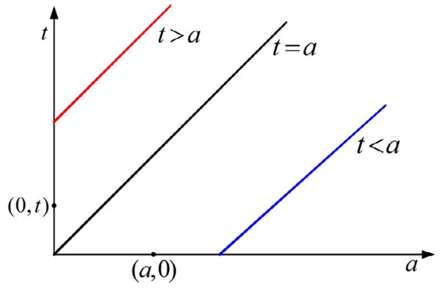

Fig. 3. Characteristic line. Red solid line is $t>a$; black solid line is $t=a$; blue solid line is $t<a$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Lemma 3.2.1 Fluctuation Lemma [40]**.** *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a bounded and continuously differentiable function. Then there exist sequences $\{s_n\}$ and $\{t_n\}$, such that*

$$\lim_{n\to\infty} s_n=\infty,\ \lim_{n\to\infty} f(s_n)=\liminf_{t\to\infty} f(t)=:f_\infty,\ \lim_{n\to\infty} f'(s_n)=0,$$
$$\lim_{n\to\infty} t_n=\infty,\ \lim_{n\to\infty} f(t_n)=\limsup_{t\to\infty} f(t)=:f^\infty,\ \lim_{n\to\infty} f'(t_n)=0.$$

Lemma 3.2.2. *[32] If $f:\mathbb{R}_+\to\mathbb{R}$ be a bounded function and $k\in L^1(\mathbb{R}_+)$, we have*

$$\liminf_{t\to\infty}\int_0^t k(\theta)f(t-\theta)\geq ||k||_1\liminf_{t\to\infty} f(t)=:||k||_1 f_\infty,$$
$$\limsup_{t\to\infty}\int_0^t k(\theta)f(t-\theta)\leq ||k||_1\limsup_{t\to\infty} f(t)=:||k||_1 f^\infty.$$

Theorem 3.2.2. *Disease-free equilibrium E^0 of system (2.11) is globally asymptotically stable if $\mathcal{R}_0<1$.*

Proof. For any solution $I_k(t,\ \cdot)$ with $I_{k0}(\cdot)\in\Gamma$, for $k\in\mathbb{N}_n$, integrating Eq. (4.2b) along the characteristic line $t-a=c$ where c is a any constant is $I_k(a+c,a)=I_k(a+\varphi,a)e^{-\mu a-\int_\varphi^a\rho(s)ds}$, for $k\in\mathbb{N}_n$, where φ is determined by the signal of c. According to Fig. 3, we get

$$\varphi=\begin{cases}0, & c>0,\\ -c, & c\leq 0,\end{cases} I_k(t,a)=\begin{cases}B_k(t-a)\pi(a), & t\geq a,\\ I_{k0}(a-t)\frac{\pi(a)}{\pi(a-t)}, & t<a,\end{cases}\tag{3.13}$$

for $k\in\mathbb{N}_n$, where $B_k(t-a)=I_k(t-a,0)=k\sigma S_k(t-a)\Theta(I_k(t-a,\cdot))+\varepsilon R_k(t-a)$. Therefore, we obtain

$$\begin{aligned}B_k(t)&=I_k(t,0)\\&=k\sigma S_k(t)\Theta(I_k(t,\cdot))+\varepsilon R_k(t)\\&=\frac{k\sigma}{\langle k\rangle}S_k(t)\sum_{k=1}^n(k-1)p(k)\int_0^\infty\beta_k(a)I_k(t,a)da+\varepsilon R_k(t),\end{aligned}\tag{3.14}$$

and $B_k(t)$ is nonnegative, bounded and differentiable for $k\in\mathbb{N}_n$. Therefore,

$$B_k(t)\leq\frac{k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\langle k\rangle\mu}\sum_{k=1}^n(k-1)p(k)\int_0^\infty\beta_k(a)I_k(t,a)da+\frac{\varepsilon\Lambda k(1-N_k^*)\varphi^*}{\mu}$$

$$
\begin{aligned}
&= \frac{k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\langle k\rangle\mu}\left(\sum_{k=1}^{n}(k-1)p(k)\left(\int_0^t \beta_k(a)B_k(t-a)\pi(a)da\right.\right.\\
&\quad\left.\left. + \int_t^\infty \beta_k(a)I_{k0}(a-t)\frac{\pi(a)}{\pi(a-t)}da\right)\right) + \frac{\varepsilon\Lambda k(1-N_k^*)\varphi^*}{\mu}\\
&\le \frac{k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\langle k\rangle\mu}\left(\sum_{k=1}^{n}(k-1)p(k)\left(\int_0^t \beta_k(a)B_k(t-a)\pi(a)da\right.\right.\\
&\quad\left.\left. + e^{-\phi t}\int_t^\infty \beta_k(a)I_{k0}(a-t)da\right)\right) + \frac{\varepsilon\Lambda k(1-N_k^*)\varphi^*}{\mu},
\end{aligned}
\tag{3.15}
$$

for $k \in \mathbb{N}_n$. Applying to Lemma (3.2.2), we obtain $B^\infty \le B^\infty A$, where

$$
B^\infty = \left(\limsup_{t\to\infty} B_k(t)\right)_{1\times n}, \tag{3.16}
$$

$$
A = \left(\frac{k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\langle k\rangle\mu}\int_0^\infty k(j-1)p(k)\beta_k(a)\pi(a)da + \frac{\varepsilon\Lambda k(j-1)(1-N_k^*)\varphi^*}{\mu}\right)_{n\times n}, \tag{3.17}
$$

for $k \in \mathbb{N}_n$. It is easy to verify that $\rho(A) = R_0 < 1, k \in \mathbb{N}_n$, where $\rho(A)$ is the spectral radius of matrix A. Thus, $B^\infty = 0$, for $k \in \mathbb{N}_n$.

First, we prove $I_k(t,\ \cdot) \to 0$ in $L_+^1[0,\infty)$ as $t \to \infty, k \in \mathbb{N}_n$. Namely,

$$
\lim_{t\to\infty}\int_0^\infty I_k(t,a)da = 0,\ \ k \in \mathbb{N}_n. \tag{3.18}
$$

Combining with Eq. (5.1), we yield

$$
\begin{aligned}
\int_0^\infty I_k(t,a)da &= \int_0^t B_k(t-a)\pi(a)da + \int_t^\infty I_{k0}(a-t)\frac{\pi(a)}{\pi(a-t)}da\\
&\le \int_0^t B_k(t-a)\pi(a)da + e^{-\phi t}\int_t^\infty I_{k0}(a-t)da,
\end{aligned}
\tag{3.19}
$$

for $k \in \mathbb{N}_n$. According to Lemma 3.2.2, we get

$$
\limsup_{t\to\infty}\int_0^\infty I_k(t,a)da \le B^\infty ||\pi||_1 \le \frac{1}{\phi}B^\infty = 0,\ \ k \in \mathbb{N}_n. \tag{3.20}
$$

Therefore, $\lim_{t\to\infty}\int_0^\infty I_k(t,a)da = 0$, for $k \in \mathbb{N}_n$.

Next, we prove that $\lim_{t\to\infty} S_k(t) = \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}$, $\lim_{t\to\infty} R_k(t) = 0$ for $k \in \mathbb{N}_n$. According to Lemma 3.2.1, there exist sequences $\{s_n\}$, $\{t_n\}$, such that

$$
\lim_{n\to\infty} s_n = \infty,\ \lim_{n\to\infty} S_k(s_n) = \liminf_{n\to\infty} S_k(t) := S_{k\infty},\ \lim_{t\to\infty} S_k'(s_n) = 0, \tag{3.21}
$$

$$
\lim_{n\to\infty} t_n = \infty,\ \lim_{n\to\infty} S_k(t_n) = \limsup_{n\to\infty} S_k(t) := S_k^\infty,\ \lim_{t\to\infty} S_k'(t_n) = 0, \tag{3.22}
$$

$$
\lim_{n\to\infty} s_n = \infty,\ \lim_{n\to\infty} R_k(s_n) = \liminf_{n\to\infty} R_k(t) := R_{k\infty},\ \lim_{t\to\infty} R_k'(s_n) = 0, \tag{3.23}
$$

$$\lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} R_k(t_n) = \limsup_{n\to\infty} R_k(t) := R_k^{\infty}, \lim_{t\to\infty} R'_k(t_n) = 0, \tag{3.24}$$

for $k \in \mathbb{N}_n$. Hence, we have

$$\lim_{n\to\infty} \int_0^{\infty} \beta_k(a) I_k(s_n, a) da = 0, \lim_{n\to\infty} \int_0^{\infty} \rho(a) I_k(s_n, a) da = 0, \tag{3.25}$$

$$\lim_{n\to\infty} \int_0^{\infty} \beta_k(a) I_k(t_n, a) da = 0, \lim_{n\to\infty} \int_0^{\infty} \rho(a) I_k(t_n, a) da = 0, \tag{3.26}$$

for $k \in \mathbb{N}_n$. From Eq. (4.2b), we obtain

$$\frac{dR_k(s_n)}{dt} = \int_0^{\infty} \rho(a) I_k(s_n, a) da - (\gamma + \mu + \varepsilon) R_k(s_n), \tag{3.27}$$

$$\frac{dR_k(t_n)}{dt} = \int_0^{\infty} \rho(a) I_k(t_n, a) da - (\gamma + \mu + \varepsilon) R_k(t_n), \tag{3.28}$$

for $k \in \mathbb{N}_n$. It follows that $0 = -(\gamma + \mu) R_{k\infty}, 0 = -(\gamma + \mu) R_k^{\infty}$ for $k \in \mathbb{N}_n$. This gives $R_{k\infty} = 0$, $R_k^{\infty} = 0$, for $k \in \mathbb{N}_n$, which indicate that $\lim_{n\to\infty} R_k(t) = 0, k \in \mathbb{N}_n$. According to Eq. (4.2a), we have

$$\frac{dS_k(s_n)}{dt} = \Lambda k\big(1 - N_k^*\big)\varphi^* + \gamma R_k(s_n) - k\sigma S_k(s_n)\Theta(I_k(s_n, \cdot)) - \mu S_k(s_n), \tag{3.29}$$

$$\frac{dS_k(t_n)}{dt} = \Lambda k\big(1 - N_k^*\big)\varphi^* + \gamma R_k(t_n) - k\sigma S_k(t_n)\Theta(I_k(t_n, \cdot)) - \mu S_k(t_n), \tag{3.30}$$

for $k \in \mathbb{N}_n$. Hence, $0 = \Lambda k\big(1 - N_k^*\big)\varphi^* - \mu S_{k\infty}$, $0 = \Lambda k\big(1 - N_k^*\big)\varphi^* - \mu S_k^{\infty}$, for $k \in \mathbb{N}_n$. It follows that $S_{k\infty} = \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}$, $S_k^{\infty} = \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}$, for $k \in \mathbb{N}_n$, which imply that $\lim_{n\to\infty} S_k(t) = \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}, k \in \mathbb{N}_n$. In a word, we have shown that

$$(S_k(t), I_k(t, \cdot), R_k(t)) \to E^0 \ in\ X\ as\ t \to \infty, k \in \mathbb{N}_n, \tag{3.31}$$

which suggests the solutions $S_k(t), \int_0^{\infty} I_k(t, a) da, R_k(t)$ of system (2.11) with initial conditions S_{k0}, $I_{k0}(\cdot)$, $R_{k0} \in \Gamma$ are attracted in Γ, for $k \in \mathbb{N}_n$. Combining with Theorem 3.2.1, disease-free equilibrium E^0 of system (2.11) is globally asymptotically stable if $\mathcal{R}_0 < 1$. This completes the proof. □

3.3. Existence of endemic equilibrium

Theorem 3.3.1. *System (2.11) has a unique positive endemic equilibrium if $R_0 > 1$.*

Proof. We assume that $E^{**} = \big\{\big(S_k^{**}, I_k^{**}(\cdot), R_k^{**}\big)\big\}_k, k \in \mathbb{N}_n$ is the endemic equilibrium of system (2.11), then we have

$$\begin{cases} 0 = \Lambda k\big(1 - N_k^*\big)\varphi^* + \gamma R_k^{**} - k\sigma S_k^{**}\Theta(I_k^{**}(\cdot)) - \mu S_k^{**}, & (3.32a)\\ \frac{dI_k^{**}(\cdot)}{da} = -\rho(a) I_k^{**}(\cdot) - \mu I_k^{**}(\cdot), & (3.32b)\\ 0 = \int_0^{\infty} \rho(a) I_k^{**}(\cdot) da - (\gamma + \mu + \varepsilon) R_k^{**}, & (3.32c) \end{cases}$$

with the boundary condition $I_k^{**}(0) \triangleq B_k^{**} = k\sigma S_k^{**}\Theta(I_k^{**}(\cdot)) + \varepsilon R_k^{**}$ for $k \in \mathbb{N}_n$. According to Eq. (3.32b), we get

$$I_k^{**}(\cdot) = I_k^{**}(0)e^{-\mu a - \int_0^a \rho(s)ds} = B_k^{**}e^{-\mu a - \int_0^a \rho(s)ds} = B_k^{**}\pi(a), \tag{3.33}$$

for $k \in \mathbb{N}_n$. Substituting it into the expression of B_k^{**}, we have

$$B_k^{**} = \frac{\Lambda k(1-N_k^*)\varphi^*\sigma k\langle k\rangle^{-1}\int_0^\infty \beta_k(a)\pi(a)da\sum_{k=1}^n(k-1)p(k)B_k^{**}}{\mu + \mu\sigma e^{-\int_0^a \phi(s)ds}k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da\sum_{k=1}^n(k-1)p(k)B_k^{**}}, \tag{3.34}$$

for $k \in \mathbb{N}_n$. Let $\sum_{k=1}^n(k-1)p(k)B_k^{**} \triangleq \tilde{B_k^{**}}, k \in \mathbb{N}_n$. Eq. (3.34) becomes

$$\tilde{B_k^{**}} = \frac{\Lambda k(1-N_k^*)\varphi^*\sigma\int_0^\infty \beta_k(a)\pi(a)da}{\mu\langle k\rangle}\sum_{k=1}^n \frac{k(k-1)p(k)\tilde{B_k^{**}}}{1 + e^{-(\mu-\varepsilon)a}\sigma k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da\tilde{B_k^{**}}}, \tag{3.35}$$

for $k \in \mathbb{N}_n$. We define the right-hand side of Eq. (3.35) as $g(\tilde{B_k^{**}})$, that is to say, $g(\tilde{B_k^{**}}) \triangleq$

$$\frac{\Lambda k(1-N_k^*)\varphi^*\sigma\int_0^\infty \beta_k(a)\pi(a)da}{\mu\langle k\rangle}\sum_{k=1}^n \frac{k(k-1)p(k)\tilde{B_k^{**}}}{1 + e^{-(\mu-\varepsilon)a}\sigma k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da\tilde{B_k^{*}}*}, \tag{3.36}$$

for $k \in \mathbb{N}_n$. Obviously, $g'(\tilde{B_k^{**}})$

$$= \frac{\Lambda k(1-N_k^*)\varphi^*\sigma\int_0^\infty \beta_k(a)\pi(a)da}{\mu\langle k\rangle}\sum_{k=1}^n \frac{k(k-1)p(k)}{(1 + e^{-(\mu-\varepsilon)a}\sigma k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da\tilde{B_k^{**}})^2}, \tag{3.37}$$

for $k \in \mathbb{N}_n$. $g''(\tilde{B_k^{**}})$

$$= -\frac{\Lambda k(1-N_k^*)\varphi^*\sigma\int_0^\infty \beta_k(a)\pi(a)da}{\mu\langle k\rangle}\sum_{k=1}^n \frac{2k(k-1)p(k)e^{-(\mu-\varepsilon)a}\sigma k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da}{(1 + e^{-(\mu-\varepsilon)a}\sigma k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da\tilde{B_k^{**}})^3}. \tag{3.38}$$

for $k \in \mathbb{N}_n$. Therefore, $g''(\tilde{B_k^{**}}) < 0$,

$$\lim_{\tilde{B_k^{**}} \to +\infty} g'(\tilde{B_k^{**}}) = 0, \tag{3.39}$$

$$\lim_{\tilde{B_k^{**}} \to 0^+} g'(\tilde{B_k^{**}}) = \frac{\Lambda k(1-N_k^*)\varphi^*\sigma\int_0^\infty \beta_k(a)\pi(a)da}{\mu\langle k\rangle}\sum_{k=1}^n k(k-1)p(k) = R_0 > 1, \tag{3.40}$$

for $k \in \mathbb{N}_n$. Then $g'(\tilde{B_k^{**}}) > 0$, $\lim_{\tilde{B_k^{**}} \to 0^+} g(\tilde{B_k^{**}}) = 0$, $\lim_{\tilde{B_k^{**}} \to +\infty} g(\tilde{B_k^{**}})$

$$= \frac{\Lambda k(1-N_k^*)\varphi^*\sigma\int_0^\infty \beta_k(a)\pi(a)da}{\mu\langle k\rangle}\sum_{k=1}^n \frac{k(k-1)p(k)}{1 + e^{-(\mu-\varepsilon)a}\sigma k\langle k\rangle^{-1}\int_0^\infty \pi(a)da\int_0^\infty \beta_k(a)\pi(a)da}, \tag{3.41}$$

for $k \in \mathbb{N}_n$. It follows that Eq. (3.35) has a unique positive solution. Due to Eqs. (3.32a) and (3.32c), we have

$$S_k^{**} = \frac{\Lambda k(1-N_k^*)\varphi^* + \gamma R_k^{**}}{\mu + k\sigma\Theta(I_k^{**}(\cdot))}, R_k^{**} = \frac{\int_0^\infty \rho(a)I_k^{**}(\cdot)da}{\gamma + \mu + \varepsilon}, \tag{3.42}$$

for $k \in \mathbb{N}_n$. Hence, system (2.11) has a unique positive endemic equilibrium, if $R_0 > 1$. This completes the proof. □

3.4. Analysis of the endemic equilibrium

3.4.1. Asymptotic smoothness

Lemma 3.4.1 [6]. *Let* $\mathbb{D}^n \subseteq \mathbb{R}^n$, *suppose*

$$|f_i(x_k)| \le M_{ki}, |f_i(x_{k1}) - f_i(x_{k2})| \le K_{ki},$$

then

$$|f_1 f_2(x_{k1}) - f_1 f_2(x_{k2})| \le K_{k1}M_{k2} + K_{k2}M_{k1},$$

$i = 1, 2,\ k \in \mathbb{N}_n$, *for* $\forall\ x_k, x_{k1}, x_{k2} \in \mathbb{D}^n$.

Definition 3.4.1. [41] A semi-flow

$$\Phi(t, x_{k0}) : \left(\mathbb{R}^+\right)^n \times X \to X, k \in \mathbb{N}_n,$$

is asymptotically smooth, if, for any nonempty, closed bounded set $A \subset X$ for which $\Phi(t, A) \subset A$, there is a compact set $A_0 \subset A$, such that A_0 attracts A.

Lemma 3.4.2 [42]. *Let* $A^n \subset (L^p[0, \infty))^n$ *be bounded and closed where* $p \ge 1$. *Then* A^n *is compact if the following conditions hold true.*

(i) $\lim_{x\to 0} \int_0^\infty |f_k(t+x) - f_k(x)|^p dt = 0$ *uniformly for* $f_k \in A^n$, $k \in \mathbb{N}_n$;
(ii) $\lim_{x\to\infty} \int_x^\infty |f_k(t)|^p dt = 0$ *uniformly for* $f_k \in A^n$, $k \in \mathbb{N}_n$.

Theorem 3.4.1. *The semi-flow*

$$\Phi(t, x_{k0}) = \varphi(t, x_{k0}) + \psi(t, x_{k0}) : \left(\mathbb{R}^+\right)^n \times X \to X, k \in \mathbb{N}_n,$$

is asymptotically smooth in X, *if the following two conditions hold.*

(i) There exists a continuous function

$$f_k : \left(\mathbb{R}^+\right)^n \times \left(\mathbb{R}^+\right)^n \to \left(\mathbb{R}^+\right)^n,$$

such that

$$\lim_{t\to\infty} f_k(t, h) = 0, ||\varphi(t, x_{k0})||_X \le f_k(t, h),$$

if $\|x_{k0}\|_X \le h$, *for* $k \in \mathbb{N}_n$;

(ii) $\psi(t, x_{k0})$ *is completely continuous, for* $t \ge 0$, $k \in \mathbb{N}_n$.

Proof. Similar to [43–45], we decompose $\Phi(t, x_{k0}) : \left(\mathbb{R}^+\right)^n \times X \to X, k \in \mathbb{N}_n$ into the following two operators $\varphi(t, x_{k0}), \psi(t, x_{k0}) : \left(\mathbb{R}^+\right)^n \times X \to X, k \in \mathbb{N}_n$, namely,

$$\varphi(t, x_{k0}) := (0, y_{1k}(t, \cdot), 0), \psi(t, x_{k0}) := \big(s_k(t), \hat{y}_{1k}(t, \cdot), r_k(t)\big), k \in \mathbb{N}_n, \tag{3.43}$$

where

$$y_{1k}(t, a) = \begin{cases} 0, & t > a \ge 0, \\ i_k(t, a), & a \ge t \ge 0, \end{cases} \quad \hat{y}_{1k}(t, a) = \begin{cases} i_k(t, a), & t > a \ge 0, \\ 0, & a \ge t \ge 0, \end{cases} \tag{3.44}$$

for $t \geq 0,\ k \in \mathbb{N}_n$.

Next, let $f_k(t,h) = he^{-(m_0+\mu)t}, h > 0, k \in \mathbb{N}_n$. It is clearly that $\lim_{t\to\infty} f_k(t,h) = 0, k \in \mathbb{N}_n$. Therefore,

$$y_{1k}(t,a) = \begin{cases} 0, & t > a \geq 0, \\ i_{k0}(a-t)\frac{\pi(a)}{\pi(a-t)}, & a \geq t \geq 0, \end{cases} \quad k \in \mathbb{N}_n. \tag{3.45}$$

For $x_{k0} \in \Gamma$ and $\|x_{k0}\|_X \leq h,\ k \in \mathbb{N}_n$, we get

$$\begin{aligned} ||\varphi(t,x_{k0})||_X &= \sum_{k=1}^n \left(|0| + \int_0^\infty |y_{1k}(t,a)|da + |0| \right) \\ &= \sum_{k=1}^n \int_t^\infty \left| i_{k0}(a-t)\frac{\pi(a)}{\pi(a-t)} \right| da \\ &= \sum_{k=1}^n \int_0^\infty \left| i_{k0}(\tau)\frac{\pi(t+\tau)}{\pi(\tau)} \right| da \\ &= \sum_{k=1}^n \int_0^\infty \left| i_{k0}(\tau) exp\left(-\int_\tau^{t+\tau} \phi(\nu)d\nu \right) \right| d\tau, \end{aligned} \tag{3.46}$$

for $k \in \mathbb{N}_n$. Note that $\phi(a) \geq m_0 + \mu, a \geq 0$, we have

$$\begin{aligned} ||\varphi(t,x_{k0})||_X &\leq e^{-(m_0+\mu)t} \sum_{k=1}^n \left(|0| + \int_0^\infty |y_{1k}(t,a)|da + |0| \right) \\ &= e^{-(m_0+\mu)t} ||x_{k0}||_X \leq he^{-(m_0+\mu)t} \triangleq f_k(t,h), \end{aligned} \tag{3.47}$$

for $k \in \mathbb{N}_n$.

Finally, according to Lemma 3.4.2, we obtain that $\psi(t, B)$ is compact, for any closed and bounded set $B \subset X$. Due to Lemma 2.2.2, we get that $S_k(t)$, $\int_0^\infty I_k(t,a)da$ and $R_k(t)$ are in the compact set $\left[0, \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}\right] \subset [0, M]$, where $M \geq \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}$ is a bound for B, $k \in \mathbb{N}_n$. Meanwhile, $\int_0^\infty \beta_k(a)I_k(t,a)da \leq \frac{\overline{\beta}\Lambda k(1-N_k^*)\varphi^*}{\mu} \leq \overline{\beta}M \triangleq L_1,\ k \in \mathbb{N}_n$, which suggests that

$$\begin{aligned} B_k(t) &= \frac{k\sigma}{\langle k \rangle} S_k(t) \sum_{k=1}^n (k-1)p(k) \int_0^\infty \beta_k(a)I_k(t,a)da \\ &\leq \frac{\Lambda k^2(1-N_k^*)\varphi^*\sigma}{\mu\langle k \rangle} \sum_{k=1}^n (k-1)p(k)nL_1 \\ &\leq k\sigma nL_1 \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \leq k\sigma MnL_1 \triangleq L_2, \end{aligned} \tag{3.48}$$

and $\Theta(I_k(t,\cdot)) = \langle k \rangle^{-1} \sum_{k=1}^n (k-1)p(k) \int_0^\infty \beta_k(a)I_k(t,a)da \leq nL_1$, then

$$\begin{aligned} \left| \frac{S_k(t)}{dt} \right| &\leq \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} + \gamma \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} + k\sigma \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} nL_1 + \mu \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \\ &= 2\Lambda k \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} + (\gamma + k\sigma nL_1) \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \\ &\leq 2\Lambda k \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} + (\gamma + k\sigma nL_1)M \triangleq L_3, \end{aligned} \tag{3.49}$$

for $t \in \mathbb{R}^+$, $(S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma_0$, $k \in \mathbb{N}_n$. Note that L_1, L_2 and L_3 are independent of t and the initial conditions. Therefore, it is shown that $\hat{y}_{1k}(t, a)$ remains in a compact subset of $\left(L_+^1[0, \infty)\right)^n$, which is independent of $x_{k0} \in \Gamma$, $k \in \mathbb{N}_n$. It is clearly that Lemma 3.4.2 holds true.

From Eqs. (5.1) and (3.44), we obtain

$$0 \le \hat{y}_{1k}(t, a) = \begin{cases} B_k(t-a)\pi(a), & t > a \ge 0, \\ 0, & a \ge t \ge 0, \end{cases} \quad k \in \mathbb{N}_n. \tag{3.50}$$

Let $t \in \mathbb{R}^+$ and $h > 0$, we have

$$\begin{aligned}
& |B_k(t+h) - B_k(t)| \\
=& \left| \frac{k\sigma}{\langle k \rangle} S_k(t+h) \sum_{k=1}^{n} (k-1)p(k) \int_0^\infty \beta_k(a) i_k(t+h, a)da + \varepsilon R_k(t+h) \right. \\
& \left. - \frac{k\sigma}{\langle k \rangle} S_k(t) \sum_{k=1}^{n} (k-1)p(k) \int_0^\infty \beta_k(a) i_k(t, a)da - \varepsilon R_k(t) \right| \\
\le& \left| \frac{k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) \left(S_k(t+h) \int_0^\infty \beta_k(a) I_k(t+h, a)da \right. \right. \\
& \left. \left. - S_k(t) \int_0^\infty \beta_k(a) I_k(t, a)da \right) \right| + |\varepsilon R_k(t+h) - \varepsilon R_k(t)| \\
\le& \frac{\varepsilon k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) \left| (S_k(t+h) - S_k(t)) \int_0^\infty \beta_k(a) I_k(t+h, a)da \right| \\
& + \frac{\varepsilon k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) \left| S_k(t) \left(\int_0^\infty \beta_k(a) I_k(t+h, a)da - \int_0^\infty \beta_k(a) I_k(t, a)da \right) \right| \\
\le& \frac{\varepsilon k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) \left(L_1 L_3 \ h + \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \int_0^h \beta_k(a) I_k(t+h, a)da \right. \\
& \left. + \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \left| \int_h^\infty \beta_k(a) I_k(t+h, a)da - \int_0^\infty \beta_k(a) I_k(t, a)da \right| \right) \\
=& \frac{\varepsilon k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) \left(L_1 L_3 \ h + \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \int_0^h \beta_k(a) B_k(t+h-a)\pi(a)da \right. \\
& \left. + \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \left| \int_h^\infty \beta_k(a) I_k(t+h, a)da - \int_0^\infty \beta_k(a) I_k(t, a)da \right| \right) \\
\le& \frac{\varepsilon k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) \left(L_1 L_3 \ h + L_1 L_2 \ h + \frac{\Lambda k(1-N_k^*)\varphi^*}{\mu} \left| \int_h^\infty \beta_k(a) I_k(t+h, a)da \right. \right. \\
& \left. \left. - \int_0^\infty \beta_k(a) I_k(t, a)da \right| \right) \\
=& \frac{\varepsilon k\sigma}{\langle k \rangle} \sum_{k=1}^{n} (k-1)p(k) (L_1 L_3 \ h + L_1 L_2 \ h
\end{aligned}$$

$$+\frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}\left|\int_0^\infty (\beta_k(a+h)I_k(t+h,a+h)-\beta_k(a)I_k(t,a))da\right|\Bigg), \tag{3.51}$$

for $k\in\mathbb{N}_n$. By Eq. (5.1), we have $I_k(t+h,a+h)=I_k(t,a)\frac{\pi(a+h)}{\pi(a)}$, $(t,h,a)\in\mathbb{R}_+^3$, $k\in\mathbb{N}_n$. Thus

$$\begin{aligned}
&\frac{\varepsilon k\sigma}{\langle k\rangle}\sum_{k=1}^n(k-1)p(k)\frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}\left|\int_0^\infty (\beta_k(a+h)I_k(t+h,a+h)-\beta_k(a)I_k(t,a))da\right|\\
=&\frac{\varepsilon k\sigma}{\langle k\rangle}\sum_{k=1}^n(k-1)p(k)\frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}\left|\int_0^\infty \left(\beta_k(a+h)I_k(t,a)\frac{\pi(a+h)}{\pi(a)}-\beta_k(a)I_k(t,a)\right)da\right|\\
\le&\frac{\varepsilon k\sigma}{\langle k\rangle}\sum_{k=1}^n(k-1)p(k)\frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}\Bigg(\int_0^\infty \beta_k(a+h)\left|I_k(t,a)\frac{\pi(a+h)}{\pi(a)}-I_k(t,a)\right|da\\
&+\int_0^\infty|\beta_k(a+h)-\beta_k(a)|I_k(t,a)da\Bigg)\\
\le&\frac{\varepsilon k\sigma}{\langle k\rangle}\sum_{k=1}^n(k-1)p(k)\frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}\Bigg(\int_0^\infty \beta_k(a+h)\left(1-e^{-\overline{\phi}h}\right)I_k(t,a)da\\
&+\int_0^\infty|\beta_k(a+h)-\beta_k(a)|I_k(t,a)da\Bigg)\\
\le&\frac{\varepsilon k\sigma}{\langle k\rangle}\sum_{k=1}^n(k-1)p(k)\frac{\Lambda k(1-N_k^*)\varphi^*}{\mu}L_1\overline{\phi}h=\frac{\varepsilon k\sigma\Lambda L_1\overline{\phi}h}{\mu}\le\varepsilon k\sigma ML_1\overline{\phi}h,
\end{aligned}\tag{3.52}$$

for $h>0$, $k\in\mathbb{N}_n$. Then $\forall\,\varepsilon>0$, $\exists\ \delta=\frac{\varepsilon'}{L_1L_2\ \ h+L_1L_3\ \ h+k\sigma ML_1\overline{\phi}h}$, such that

$$|B_k(t+h)-B_k(t)|\le\varepsilon',t\in\mathbb{R}^+,h\in(0,\delta),(S_{k0},I_{k0}(\cdot),R_{k0})\in\Gamma_0,k\in\mathbb{N}_n. \tag{3.53}$$

Furthermore, we have

$$\begin{aligned}
\int_0^\infty\left|\hat{y}_{1k}(t,a+h)-\hat{y}_{1k}(t,a)\right|da=&\int_0^t|I_k(t,a+h)-I_k(t,a)|da\\
=&\int_0^{t-h}|B_k(t-a-h)\pi(a+h)-B_k(t-a)\pi(a)|da\\
&+\int_{t-h}^t B_k(t-a)\pi(a)da\\
\le&\int_0^{t-h}|B_k(t-a-h)-B_k(t-a)|\pi(a)da\\
&+\int_0^{t-h}B_k(t-a-h)|\pi(a+h)-\pi(a)|da+L_2h\\
\le&(t-h)L_2\ \ h\overline{\phi}+\int_0^{t-h}|B_k(t-a-h)-B_k(t-a)|\\
&\times\ \pi(a)da+L_2\ \ h,
\end{aligned}\tag{3.54}$$

for $h>0$, $k\in\mathbb{N}_n$. Combining with Eqs. (3.51) and (3.54), we have

$$\lim_{h\to0}\int_0^\infty\left|\hat{y}_{1k}(t,a+h)-\hat{y}_{1k}(t,a)\right|da=0, \tag{3.55}$$

then $\forall x_{k0} \in B$, $\hat{y}_{1k}(t, a)$ remains in a compact subset $B_{\hat{y}_{1k}}$ of $L^1_+[0, \infty)$, $k \in \mathbb{N}_n$. Thus, $\psi(t, B) \subseteq [0, M] \times B_{\hat{y}_{1k}} \times [0, M]$, which is compact in X, $k \in \mathbb{N}_n$. Then, $\psi(t, x_{k0})$, $k \in \mathbb{N}_n$ is completely continuous. This completes the proof. □

3.4.2. Uniform permanence

In this section, we build the attractivity of the endemic equilibrium E^{**} by using Lyapunov functional. Therefore, we need the following permanence of system (2.11). According to [46], we introduce the following concepts. Define $\rho : \Gamma \to \left(\mathbb{R}^+\right)^n$, with

$$\rho(S_1, S_2, \dots, S_n; I_1(a), I_2(a), \dots, I_n(a); R_1, R_2, \dots, R_n) = \frac{\sum_{k=1}^n (k-1)p(k) \int_0^\infty \beta_k(a) B_k^{**} \pi(a) da}{\langle k \rangle}, \tag{3.56}$$

for $(S_1, S_2, \dots, S_n; I_1(a), I_2(a), \dots, I_n(a); R_1, R_2, \dots, R_n) \in \Gamma$. Meanwhile,

$$\Gamma_0 = \left\{(S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma : \exists\ t_0 \in \mathbb{R}^+, s.t.\ \rho(\Phi(t_0, (S_{k0}, I_{k0}(a), R_{k0})) > 0, k \in \mathbb{N}_n\right\}. \tag{3.57}$$

Easily, if $(S_k(t), I_k(t, \cdot), R_k(t)) \in \Gamma \backslash \Gamma_0, k \in \mathbb{N}_n$, $(S_k(t), I_k(t, \cdot), R_k(t)) \to E^{**}$, as $t \to \infty, k \in \mathbb{N}_n$.

Definition 3.4.2. For system (2.11), we obtain

(i) If $\exists\, \eta > 0$, independent of the initial conditions, such that

$$\limsup_{t \to \infty} \rho(\Phi(t, (S_{k0}, I_{k0}(a), R_{k0})) > \eta,$$

system (2.11) is *uniformly weakly ρ-persistent*, for $(S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma_0$, $\Theta(I_{k0}(\cdot)) > 0$ and $k \in \mathbb{N}_n$;

(ii) if $\exists\, \eta > 0$, independent of the initial conditions, such that

$$\liminf_{t \to \infty} \rho(\Phi(t, (S_{k0}, I_{k0}(a), R_{k0})) > \eta,$$

system (2.11) is *uniformly strongly ρ-persistent*, for $(S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma_0$, $\Theta(I_{k0}(\cdot)) > 0$ and $k \in \mathbb{N}_n$.

Theorem 3.4.2. *If* $R_0 > 1$, *system (2.11) is uniformly weakly ρ-persistent, for* $(S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma_0$, $\Theta(I_{k0}(\cdot)) > 0$ *and for* $k \in \mathbb{N}_n$.

Proof. Due to $R_0 > 1$, we find an $2\eta_0 > 0$, such that

$$\left(\frac{\Lambda k(1 - N_k^*)\varphi^*}{2nk\sigma\eta_0\mu} - 2\eta_0\right)\frac{k\sigma}{\langle k \rangle}\sum_{k=1}^n (k-1)p(k)\int_0^\infty \beta_k(a)\pi(a)e^{-2\eta_0 a}da > 1, k \in \mathbb{N}_n. \tag{3.58}$$

We assume, by way of contradiction, that system (2.11) is not uniformly weakly ρ-persistence. Then $\exists (S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma_0$ and $\eta_0 > 0$, such that

$$\limsup_{t \to \infty} \rho(\Phi(t, (S_{k0}, I_{k0}(a), R_{k0}))) \le \eta_0, k \in \mathbb{N}_n. \tag{3.59}$$

Furthermore, $\exists\ t_0 \in \mathbb{R}^+$, such that $\rho(\Phi(t, (S_{k0}, I_{k0}(a), R_{k0}))) \le 2\eta_0, for\ t \ge t_0, k \in \mathbb{N}_n$. Without loss of generality, we suppose $t_0 = 0$. We achieve it by replacing the initial condition with $\Phi(t, (S_{k0}, I_{k0}(a), R_{k0}))$, for $k \in \mathbb{N}_n$. Thus, if $t \ge t_0 = 0$, we get

$$B_k(t) = \frac{k\sigma}{\langle k \rangle} S_k(t) \sum_{k=1}^n (k-1)p(k) \int_0^\infty \beta_k(a) I_k(t, a) da$$

$$
\begin{aligned}
&\leq \frac{k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\mu\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\int_0^\infty \beta_k(a)I_k(t,a)da\\
&\leq \frac{2\eta_{k0}k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\mu\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\\
&= \frac{2n\eta_0 k^2\sigma\Lambda(1-N_k^*)\varphi^*}{\mu},
\end{aligned}
\tag{3.60}
$$

for $k\in\mathbb{N}_n$. According to Eq. (4.2a), we have

$$
\begin{aligned}
\frac{dS_k(t)}{dt} &= \Lambda k(1-N_k^*)\varphi^* + \gamma R_k(t) - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t)\\
&\geq \Lambda k(1-N_k^*)\varphi^* - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t)\\
&\geq \Lambda k(1-N_k^*)\varphi^* - (2nk\sigma\eta_0+\mu)S_k(t),
\end{aligned}
\tag{3.61}
$$

for $k\in\mathbb{N}_n$, which indicates that $\liminf_{t\to\infty} S_k(t)\geq \frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}$, $k\in\mathbb{N}_n$. Then $\exists t_1\geq t_0$, such that $S_k(t)\geq \frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}-2\eta_0$, for $t\geq t_1, k\in\mathbb{N}_n$. Again, by replacing the initial conditions, we suppose that $S_k(t)\geq \frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}-2\eta_0$, for $t\in\mathbb{R}^+, k\in\mathbb{N}_n$. Combining with Eq. (5.1), we obtain

$$
\begin{aligned}
B_k(t) &= \frac{k\sigma}{\langle k\rangle}S_k(t)\sum_{k=1}^{n}(k-1)p(k)\int_0^\infty \beta_k(a)I_k(t,a)da\\
&\geq \left(\frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}-2\eta_0\right)\frac{k\sigma}{\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\int_0^\infty \beta_k(a)I_k(t,a)da\\
&\geq \left(\frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}-2\eta_0\right)\frac{k\sigma}{\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\int_0^t \beta_k(a)I_{k0}(t-a)\frac{\pi(a)}{\pi(a-t)}da\\
&\quad+\left(\frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}-2\eta_0\right)\frac{k\sigma}{\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\int_t^\infty \beta_k(a)B_k(t-a)\pi(a)da\\
&= \left(\frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0+\mu}-2\eta_0\right)\frac{k\sigma}{\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\left(\int_0^t \beta_k(a)I_{k0}(t-a)\frac{\pi(a)}{\pi(a-t)}da\right.\\
&\quad\left.+\int_t^\infty \beta(a)B_k(t-a)\pi(a)da\right),
\end{aligned}
\tag{3.62}
$$

for $k\in\mathbb{N}_n$. Note that both $B_k(\cdot)$ and $\beta(\cdot)\pi(\cdot)$ are bounded functions on $\mathbb{R}^+$. Therefore, due to Eq. (3.62), their Laplace transforms on $\mathbb{R}^+$ are

$$
\widehat{B_k(\lambda)}\geq \left(\frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0}-2\eta_0\right)\frac{k\sigma}{\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\left(\widehat{B_k(\lambda)}\widehat{\beta(\lambda)}+\widehat{B_k(\lambda)}\widehat{\beta(\lambda)}\widehat{\pi(\lambda)}\right),
\tag{3.63}
$$

for $\lambda>0, k\in\mathbb{N}_n$, where $\widehat{\cdot}$ means the Laplace transform of a function. As $B_k(\cdot)$ is not identical zero on $\mathbb{R}^+$, we get $\widehat{B_k(\lambda)}>0, \lambda>0, k\in\mathbb{N}_n$. From Eq. (3.63), it follows that

$$
\left(\frac{\Lambda k(1-N_k^*)\varphi^*}{2nk\sigma\eta_0}-2\eta_0\right)\frac{k\sigma}{\langle k\rangle}\sum_{k=1}^{n}(k-1)p(k)\left(\widehat{\beta(\lambda)}+\widehat{\beta(\lambda)}\widehat{\pi(\lambda)}\right)\leq 1, \lambda>0, k\in\mathbb{N}_n.
\tag{3.64}
$$

It is clearly a contradiction with Eq. (3.58) by taking $\lambda = \eta_0$. This completes the proof. □

According to Lemma 2.2.2, Definition 3.4.1 and Theorem 3.4.1, we clearly obtain following result.

Theorem 3.4.3. *There exists a global attractor $\mathcal{A}$ for the solution semi-flow Φ of system (2.11) in Γ_0, if $R_0 > 1$.*

With the assistance of Theorems 3.4.2, 3.4.3 and 3.2.2 of [47], we get following Theorem 3.4.4.

Theorem 3.4.4. *If $R_0 > 1$, system (2.11) is uniformly strongly ρ-persistent, for $(S_{k0}, I_{k0}(a), R_{k0}) \in \Gamma_0$, $\Theta(I_{k0}(\cdot)) > 0$ and for $k \in \mathbb{N}_n$.*

Since the global attractor $\mathcal{A}$ is invariant, it contains all points with total trajectories through them. A total trajectory of Φ is a function $X : \mathbb{R}^n \to (\mathbb{R}^+)^n \times (L^1_+[0, \infty))^n$, such that $\Phi(s, X(t)) = X(t+s)$ for $t \in \mathbb{R}^n$ and $s \in (\mathbb{R}^+)^n$. For a total trajectory, $I_k(t, a) = I_k(t-a)\pi(a)$, for $t \in \mathbb{R}^n$ and $a \in (\mathbb{R}^+)^n$. The α limit set of a total trajectory $X(t)$ passing through $X(0) = X_0$ is

$$\alpha(X_0) = \bigcap_{t\le 0} \overline{\bigcup_{s\le t} X(S)} \subseteq \mathcal{A} \bigcap \Gamma_0, k \in \mathbb{N}_n.$$

Corollary 3.4.1. *$(S_k(t), I_k(t, a), R_k(t))$ is a total trajectory in $\mathcal{A}$, if $R_0 > 1$, $k \in \mathbb{N}_n$.*

Proof. According to Eq. (4.2a), we have

$$\begin{aligned}\frac{dS_k(t)}{dt} &\ge \Lambda k(1-N_k^*)\varphi^* - k\sigma S_k(t)\Theta(I_k(t,\cdot)) - \mu S_k(t)\\ &= \Lambda k(1-N_k^*)\varphi^* - k\sigma(nL_1+\mu)S_k(t),\end{aligned} \tag{3.65}$$

which implies $\liminf_{t\to\infty} S_k(t) \ge \frac{\Lambda k(1-N_k^*)\varphi^*}{k\sigma(nL_1+\mu)} := \varepsilon_1, k \in \mathbb{N}_n$. By Theorem 3.4.4, $\exists \varepsilon_2 > 0$, such that $\varepsilon_2 < \rho(S_k(t), I_k(t,\cdot), R_k(t)), t \in \mathbb{R}, k \in \mathbb{N}_n$. Then

$$\begin{aligned}I_k(t,0) &= \frac{k\sigma}{\langle k\rangle} S_k(t) \sum_{k=1}^{n} (k-1)p(k) \int_0^\infty \beta_k(a) I_k(t,a)da\\ &\ge \varepsilon_1 \frac{k\sigma}{\langle k\rangle} \sum_{k=1}^{n} (k-1)p(k) \int_0^\infty \beta_k(a) I_k(t,a)da\\ &\ge \varepsilon_1\varepsilon_2 \frac{k\sigma}{\langle k\rangle} \sum_{k=1}^{n} (k-1)p(k) = \varepsilon_1\varepsilon_2 k\sigma, k \in \mathbb{N}_n.\end{aligned} \tag{3.66}$$

From Eq. (4.2c), we obtain

$$\frac{dR_k(t)}{dt} \ge \frac{\overline{\rho}\Lambda k(1-N_k^*)\varphi^*}{\mu} - (\gamma+\mu+\varepsilon)R_k(t), \ \ k \in \mathbb{N}_n, \tag{3.67}$$

which implies $\liminf_{t\to\infty} R_k(t) \ge \frac{\overline{\rho}\Lambda k(1-N_k^*)\varphi^*}{(\gamma+\mu+\varepsilon)\mu} := \varepsilon_3, k \in \mathbb{N}_n$. Therefore, we take $\varepsilon_0 = (\varepsilon_1, \varepsilon_1\varepsilon_2 k\sigma, \varepsilon_3)$, such that $S_k(t), I_k(t,0), R_k(t) > \varepsilon_0, t \in \mathbb{R}, k \in \mathbb{N}_n$. This completes the proof. □

3.4.3. Stability of endemic equilibrium

Theorem 3.4.5. *Endemic equilibrium E^{**} of system (2.11) is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof. The characteristic equation of system (2.11) around endemic equilibrium $E^{**} = \left\{\left(s_k^{**}, i_k^{**}(\cdot), r_k^{**}\right)\right\}_k$, $k \in \mathbb{N}_n$ is

$$\begin{cases} \frac{dS_k(t)}{dt} = -\mu S_k(t) - k\sigma S_k^{**}\Theta(I_k(t,\cdot)) - k\sigma S_k(t)\Theta(I_k^{**}) + \gamma R_k(t), & (3.68a)\\ \frac{\partial I_k(t,a)}{\partial t} + \frac{\partial I_k(t,a)}{\partial a} = -\rho(a)I_k(t,a) - \mu I_k(t,a), & (3.68b)\\ \frac{dR_k(t)}{dt} = \int_0^\infty \rho(a)I_k(t,a)da - (\gamma+\mu+\varepsilon)R_k(t), & (3.68c) \end{cases}$$

with the boundary condition $I_k(t,0) = k\sigma S_k^{**}\Theta(I_k(t,\cdot)) + k\sigma S_k(t)\Theta(I_k^{**}) + \varepsilon R_k(t)$ for $k \in \mathbb{N}_n$. Substituting $S_k(t) = S_{k0}e^{\lambda t}, I_k(t,a) = I_k(a)e^{\lambda t}, R_k(t) = R_{k0}e^{\lambda t}, k \in \mathbb{N}_n$ into system (3.68), we have

$$\begin{cases} 0 = -(\lambda+\mu+k\sigma\Theta(I_k^{**}))S_{k0} - k\sigma S_k^{**}\Theta(I_k(\cdot)) + \gamma R_{k0}, & (3.69a)\\ \frac{dI_k(a)}{da} = -(\lambda+\rho(a)+\mu)I_k(a), & (3.69b)\\ 0 = \int_0^\infty \rho(a)I_k(a)da - (\lambda+\gamma+\mu+\varepsilon)R_{k0}, & (3.69c) \end{cases}$$

with the boundary condition $I_k(0) = k\sigma S_k^{**}\Theta(I_k(\cdot)) + k\sigma S_k(t)\Theta(I_k^{**}) + \varepsilon R_k(t)$, for $k \in \mathbb{N}_n$. Therefore, the characteristic equation of system (2.11) is

$$\begin{vmatrix} B & D & -\gamma E_n \\ E & C & 0 \\ 0 & 0 & (\lambda+\gamma+\mu+\varepsilon)E_n \end{vmatrix} = 0, \tag{3.70}$$

where

$$B = \begin{vmatrix} \lambda+\mu+b_1 & 0 & \dots & 0 \\ 0 & \lambda+\mu+2b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda+\mu+nb_n \end{vmatrix}, E = \begin{vmatrix} -b_1 & 0 & \dots & 0 \\ 0 & -2b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -nb_n \end{vmatrix},$$

$$D = \begin{vmatrix} 0 & \frac{s_1^{**}p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \dots & \frac{(n-1)s_1^{**}p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \\ 0 & \frac{2s_2^{**}p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \dots & \frac{2(n-1)s_2^{**}p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{ns_2^{**}p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \dots & \frac{n(n-1)s_n^{**}p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \end{vmatrix},$$

$$C = \begin{vmatrix} 1 & -\frac{s_1^{**}p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \dots & -(n-1)\frac{s_1^{**}p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \\ 0 & 1-\frac{2s_2^{**}p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \dots & -2(n-1)\frac{s_n^{**}p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -\frac{ns_2^{**}p(2)\int_0^\infty \beta_2(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} & \dots & 1-\frac{n(n-1)s_n^{**}p(n)\int_0^\infty \beta_n(a)\pi(a)e^{-\lambda a}da}{\langle k\rangle} \end{vmatrix},$$

where $b_k = \frac{k\sum_{j=1}^{n}(j-1)p(j)\int_0^\infty \beta_j(a)\pi(a)B_j^{**}}{\langle k\rangle}$, $k, j \in \mathbb{N}_n$. It is clear that Eq. (3.70) has n negative eigenvalue $-(\gamma+\mu+\varepsilon)$. By simple calculation, we have

$$\begin{vmatrix} B & D \\ E & C \end{vmatrix} = \prod_{k=1}^{n}(\lambda+\mu+kb_k)\left(\sum_{j=1}^{n}\left(\frac{kb_k}{\lambda+\mu+kb_k}-1\right)j(j-1)p(j)s_j^{**}\int_0^\infty \beta_j(a)\pi(a)e^{-\lambda a}da+1\right), \quad (3.71)$$

for $k, j \in \mathbb{N}_n$. Therefore, $\lambda_k = -\mu - kb_k < 0, k \in \mathbb{N}_n$, and the other characteristic roots are determined by

$$\sum_{j=1}^{n}\left(1-\frac{kb_k}{\lambda+\mu+kb_k}\right)j(j-1)p(j)s_j^{**}\int_0^\infty \beta_j(a)\pi(a)e^{-\lambda a}da = 1, \;\; k, j \in \mathbb{N}_n. \quad (3.72)$$

Combining with Eq. (3.42), we obtain

$$1 = \frac{\sum_{j=1}^{n}\left(1-\frac{kb_k}{\lambda+\mu+kb_k}\right)(j-1)p(j)\left(j\sigma\Theta(i_j^{**}(\cdot))+\mu\right)\int_0^\infty \beta_j(a)\pi(a)e^{-\lambda a}da}{\Lambda k(1-N_k^*)\varphi^*} \triangleq f(\lambda), \quad (3.73)$$

for $k, j \in \mathbb{N}_n$. We suppose that Eq. (3.71) has a characteristic root λ_0 with $\lambda_0 \geq 0$. Thus, $|f(\lambda_0)| \leq |f(Re\lambda_0)| \leq |f(0)| < 1$. This leads to a contradiction with $f(\lambda_0) = 1$. Therefore the endemic equilibrium E^{**} of system (2.11) is locally asymptotically stable. This completes the proof. □

Theorem 3.4.6. *If $B = (\beta_{ij})$, $i, j \in \mathbb{N}_n$ is irreducible, endemic equilibrium E^{**} of system (2.11) is globally asymptotically stable when $\mathcal{R}_0 > 1$.*

Proof. From Lemmas 2.2.1 and 2.2.2, we know there exist δ_1, δ_2 such that

$$\delta_1 < \frac{S_k(t)}{S_k^{**}} < \delta_2, \delta_1 < \frac{I_k(t,a)}{I_k^{**}(a)} < \delta_2, \delta_1 < \frac{R_k(t)}{R_k^{**}} < \delta_2, \;\; k \in \mathbb{N}_n. \quad (3.74)$$

It follows that $f\left(\frac{S_k(t)}{S_k^{**}}\right)$, $f\left(\frac{I_k(t,a)}{I_k^{**}(a)}\right)$, $f\left(\frac{R_k(t)}{R_k^{**}}\right)$, $k \in \mathbb{N}_n$ are bounded. We introduce Volterra-type function $f(x) = x - 1 - lnx$, then $f(x) \geq 0$ for $x > 0$ and $f'(x) = 1 - \frac{1}{x}$. Hence, $f(x)$ has a global minimum at $x = 1$ and $f(1) = 0$. Namely, $f(x)$ is nonnegative. We define

$$V_k(t) = S_k^{**} f\left(\frac{S_k(t)}{S_k^{**}}\right) + \sigma k S_k^{**}\langle k\rangle^{-1}\sum_{j=1}^{n}(j-1)p(j)\int_0^\infty \theta_j(a) f\left(\frac{I_k(t,a)}{I_k^{**}(a)}\right)da + R_k^{**} f\left(\frac{R_k(t)}{R_k^{**}}\right), \quad (3.75)$$

where $\theta_k(a) = \int_a^\infty \beta_k(s)I_k^{**}(s)ds$, then $\theta_k'(a) = -\beta_k(a)I_k^{**}(a)$, $k \in \mathbb{N}_n$. According to [43] and differentiating it, we have

$$\begin{aligned} \frac{dV_k(t)}{dt} =& \left(1-\frac{S_k^{**}}{S_k(t)}\right)\frac{dS_k(t)}{dt} + \left(1-\frac{R_k^{**}}{R_k(t)}\right)\frac{dR_k(t)}{dt} \\ &+ \sigma k S_k^{**}\langle k\rangle^{-1}\sum_{j=1}^{n}(j-1)p(j)\int_0^\infty \theta_j(a)\frac{\partial f\left(\frac{I_k(t,a)}{I_k^{**}(a)}\right)}{\partial t}da \\ =& -\frac{\mu}{S_k(t)}(S_k(t)-S_k^{**})^2 \end{aligned}$$

$$
\begin{aligned}
&+\sigma k S_k^{**}\langle k\rangle^{-1}\sum_{j=1}^{n}(j-1)p(j)\int_0^{\infty}\beta_j(a)I_j^{**}(a)\left(1-\frac{S_k^{**}}{S_k(t)}-\frac{S_k^{**}I_j(t,a)}{S_k(t)I_j^{**}(a)}+\frac{I_j(t,a)}{I_j^{**}(a)}\right)da\\
&+\sigma k S_k^{**}\langle k\rangle^{-1}\sum_{j=1}^{n}(j-1)p(j)\int_0^{\infty}\beta_j(a)I_j^{**}(a)\left(f(\frac{I_k(t,0)}{I_k^{**}(0)})-f(\frac{I_k(t,a)}{I_k^{**}(a)})\right)da\\
&-\frac{\gamma+\mu+\varepsilon}{R_k(t)}(R_k(t)-R_k^{**})^2\\
=&-\frac{\mu}{S_k(t)}(S_k(t)-S_k^{**})^2\sigma k S_k^{**}\langle k\rangle^{-1}\sum_{j=1}^{n}(j-1)p(j)\int_0^{\infty}\beta_j(a)I_j^{**}(a)\\
&\times\left(1-\frac{S_k^{**}}{S_k(t)}+ln\frac{I_k^{**}(0)}{I_k(t,0)}+\frac{I_j(t,a)}{I_j^{**}(a)}-\frac{I_k(t,a)}{I_k^{*}(a)}+ln\frac{I_k^{*}(t,a)}{I_k(a)}\right)da\\
&-\frac{\gamma+\mu+\varepsilon}{R_k(t)}(R_k(t)-R_k^{**})^2, \qquad (3.76)
\end{aligned}
$$

where $R_k^{**}=N_k^{*}-S_k^{**}-\int_0^{\infty}I_k^{**}(a)da,\ k\in\mathbb{N}_n$. Due to Corollary 3.4.1, we yield

$$
\begin{aligned}
\frac{dV_k(t)}{dt}=&\sum_{j=1}^{n}kS_k^{**}\langle k\rangle^{-1}\sum_{j=1}^{n}(j-1)p(j)\int_0^{\infty}\beta_j(a)I_j^{**}(a)\left(-f\left(\frac{S_k^{**}}{S_k(t)}\right)\right.\\
&\left.-f\left(\frac{S_k(t)I_j(t,a)I_k^{**}(0)}{S_k^{**}I_j^{**}(a)I_k(t,0)}\right)+\frac{I_k(t,a)}{I_k^{**}(a)}-ln\frac{I_k(t,a)}{I_k^{**}(a)}+\frac{I_j(t,a)}{I_j^{**}(a)}-ln\frac{I_j(t,a)}{I_j^{**}(a)}\right)da,
\end{aligned}
\qquad (3.77)
$$

for $j,k\in\mathbb{N}_n$. According to [16,24,48,49], it is verify that $\frac{dV_k(t)}{dt}$ satisfy the conditions of Theorem 3.1 and Corollary 3.3 in [16]. Furthermore, it is clear that $\exists\ c_k>0, k\in\mathbb{N}_n$, such that $V=\sum_{k=1}^{n}c_kV_k>0$ which is a Lyapunov function of system (2.11). It is easy that $V'(t)\le 0$, moreover, the largest invariant set for $V'(t)=0$ is E^{**}. Due to [50], the positive solution of system (2.11) is globally asymptotically stable. This completes the proof. □

4. Immunization strategies

Immunization is very important in controlling diseases. In [51,52], the authors discussed some immunization strategies. Therefore, in this section we discuss two immunization strategies.

4.1. Proportional immunization

Let $\omega(0<\omega<1)$ be the density of immune nodes in the network, and we substitute $\beta_k(a)(1-\omega)$ to $\beta_k(a)$. Thus, system (2.11) becomes

$$
\begin{cases}
\frac{dS_k(t)}{dt}=\Lambda k\big(1-N_k^{*}\big)\varphi^{*}+\gamma R_k(t)-k\sigma S_k(t)\Theta(I_k(t,\cdot))-\mu S_k(t), & (4.1a)\\
\frac{\partial I_k(t,a)}{\partial t}+\frac{\partial I_k(t,a)}{\partial a}=-\rho(a)I_k(t,a)-\mu I_k(t,a), & (4.1b)\\
\frac{dR_k(t)}{dt}=\int_0^{\infty}\rho(a)I_k(t,a)da-(\gamma+\mu+\varepsilon)R_k(t), & (4.1c)
\end{cases}
$$

with the boundary condition $I_k(t,0)=k\sigma S_k(t)\Theta(I_k(t,\cdot))+\varepsilon R_k(t)$ for $t\ge 0$ and the initial conditions $S_k(0)=S_{k0}, I_k(0,a)=I_{k0}(a), R_k(0)=R_{k0}$ for $a\ge 0$ and $k\in\mathbb{N}_n$. Here, $S_{k0}\in(0,$

∞), $R_{k0} \in [0, \infty)$ and $I_{k0}(a) \in L^1_+[0, \infty)$, where $L^1_+[0, \infty)$ is the space of function on $[0, \infty)$ that are nonnegative and Lebesgue integrable, and

$$\Theta(I_k(t, \cdot)) = \frac{\sum_{k=1}^{n} (k-1)p(k) \int_0^\infty \beta_k(a)(1-\omega)I_k(t, a)da}{\langle k \rangle}, \ \ k \in \mathbb{N}_n.$$

Therefore R_0 becomes

$$R_0^* = \frac{\sigma \sum_{j=1}^{n} j(j-1)S_j^0 p(j) \int_0^\infty \beta_j(a)(1-\omega)\pi(a)da}{\langle k \rangle}, k \in \mathbb{N}_n.$$

That is

$$R_0^* = (1-\omega)R_0.$$

Therefore, when $\omega = 0$, namely, $R_0^* = R_0$, no immunization schemes are done; when $0 < \omega < 1$, that is, $R_0^* < R_0$, the immunization schemes are done and effective; when $\omega = 1$, that is, $R_0^* = 0$, A full immunization schemes are done, that is to say, it would be impossible for the epidemic to spread in the network.

4.2. Targeted immunization

We introduce an upper threshold κ, such that all nodes with connectivity $k > \kappa$ are immunized. Therefore, define the immunization rate ω_k:

$$\omega_k = \begin{cases} 1, & k > \kappa, \\ c, & k = \kappa, \\ 0, & k < \kappa, \end{cases}$$

where $0 < c < 1$, and $\bar{\omega} = \sum_{k=1}^{n} \omega_k p(k)$ is the average immunization rate. Therefore, system (2.11) becomes

$$\begin{cases} \frac{dS_k(t)}{dt} = \Lambda k\big(1 - N_k^*\big)\varphi^* + \gamma R_k(t) - k\sigma S_k(t)\Theta(I_k(t, \cdot)) - \mu S_k(t), & (4.2a) \\ \frac{\partial I_k(t,a)}{\partial t} + \frac{\partial I_k(t,a)}{\partial a} = -\rho(a)I_k(t, a) - \mu I_k(t, a), & (4.2b) \\ \frac{dR_k(t)}{dt} = \int_0^\infty \rho(a)I_k(t, a)da - (\gamma + \mu + \varepsilon)R_k(t), & (4.2c) \end{cases}$$

with the boundary condition $I_k(t, 0) = k\sigma S_k(t)\Theta(I_k(t, \cdot)) + \varepsilon R_k(t)$ for $t \geq 0$ and the initial conditions $S_k(0) = S_{k0}, I_k(0, a) = I_{k0}(a), R_k(0) = R_{k0}$ for $a \geq 0$ and $k \in \mathbb{N}_n$. Here, $S_{k0} \in (0, \infty)$, $R_{k0} \in [0, \infty)$ and $I_{k0}(a) \in L^1_+[0, \infty)$, where $L^1_+[0, \infty)$ is the space of function on $[0, \infty)$ that are nonnegative and Lebesgue integrable, and

$$\Theta(I_k(t, \cdot)) = \frac{\sum_{k=1}^{n} (k-1)p(k) \int_0^\infty \beta_k(a)(1-\omega_k)I_k(t, a)da}{\langle k \rangle}, \ \ k \in \mathbb{N}_n.$$

Therefore R_0 becomes

$$R_0^{**} = \frac{\sigma \sum_{j=1}^{n} j(j-1)S_j^0 p(j) \int_0^\infty \beta_j(a)\big(\langle k^2 \rangle - \langle k^2\omega_k \rangle\big)\pi(a)da}{\langle k \rangle}, k \in \mathbb{N}_n,$$

where

$$\langle k^2\omega_k \rangle = \bar{\omega}\langle k^2 \rangle + \langle(\omega_k - \bar{\omega}) \times \big[k^2 - \langle k^2 \rangle\big]\omega_k\rangle,$$

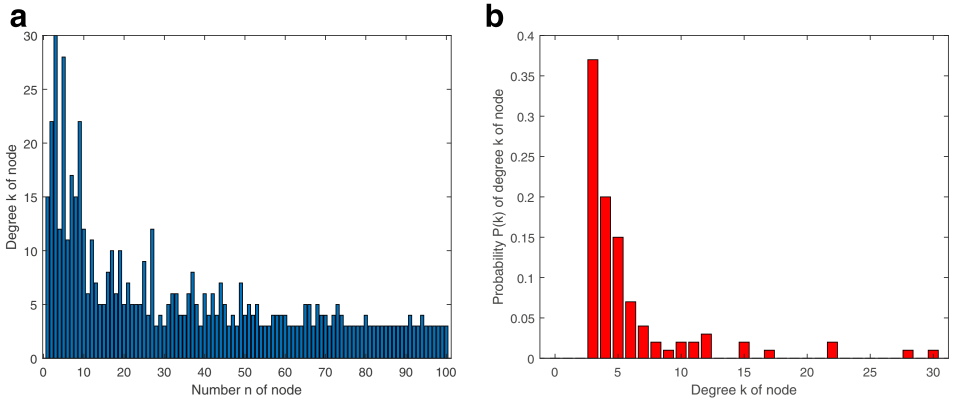

Fig. 4. (a) Size distribution of degree k of node on a scale-free network with 100 nodes and $m = 3$; (b) Probability distribution of degree k of node on a scale-free network with 100 nodes and $m = 3$.

and $\langle(\omega_k - \bar{\omega}) \times \left[k^2 - \langle k^2\rangle\right]\omega_k\rangle$ is the covariance of ω_k and k^2. Therefore, there exists a big enough κ, such that $\langle(\omega_k - \bar{\omega}) \times \left[k^2 - \langle k^2\rangle\right]\omega_k\rangle < 0$; there exists an appropriate small κ, such that $\omega_k - \bar{\omega}$ and $k^2 - \langle k^2\rangle$ have the same signs except for $\omega_k - \bar{\omega}$ or $k^2 - \langle k^2\rangle$ is 0; there exists an appropriate κ, such that $\langle(\omega_k - \bar{\omega}) \times \left[k^2 - \langle k^2\rangle\right]\omega_k\rangle > 0$. Then

$$R_0^{**} < \frac{1 - \bar{\omega}}{1 - \omega} R_0^*.$$

If we set $\bar{\omega} = \omega$,

$$R_0^{**} < R_0^* (0 < \omega < 1),$$

which indicates the targeted immunization scheme is more efficient than the uniform immunization scheme for the same average immunization rate.

5. Numerical simulation

In this section, we present some numerical results of system (2.11) that support and extend our theoretical results by using Matlab. Simulations are based on a scale-free network [53–56] with $P(k) = (r-1)m^{(r-1)}k^{-r}$, where m represents the smallest degree on a scale-free network nodes; r is variable of power law exponent. Let $m = 3,\ r = 3$ and the number of nodes on a scale-free network is $N = 100$, and we add each new node with 3 new edges (see Figs. 4 and 5). We employ iteration method and difference method to calculate density of S_k, I_k, R_k for different degree k as $k_{\min} = 3,\ k = 4,\ k = 5,\ k = 6,\ k = 7,\ k = 8,\ k = 9,\ k = 10,\ k = 11,\ k = 12,\ k = 15,\ k = 17,\ k = 22,\ k = 28,\ k_{\max} = 30$. Meanwhile, we choose some other parameters based on Table 1.

Firstly, we select a set of parameters as following:

$$\Lambda = 0.4\,\text{year}^{-1}, \mu = 0.04\,\text{year}^{-1}, \sigma = 0.08\,\text{year}^{-1}, \varepsilon = 0.6\,\text{year}^{-1}, \gamma = 0.05\,\text{year}^{-1},$$
$$r = 2.4\,\text{year}^{-1}, \beta = 0.002\,\text{year}^{-1}, \alpha = 0.6\,\text{year}^{-1}, \tau = 5\,\text{year}^{-1}, \rho = 0.01\,\text{year}^{-1}.$$

Then $R_0 = 0.8169 < 1$. Due to Theorem 3.2.2, the disease-free equilibrium E^0 is globally asymptotically stable, as is shown Fig. 6.

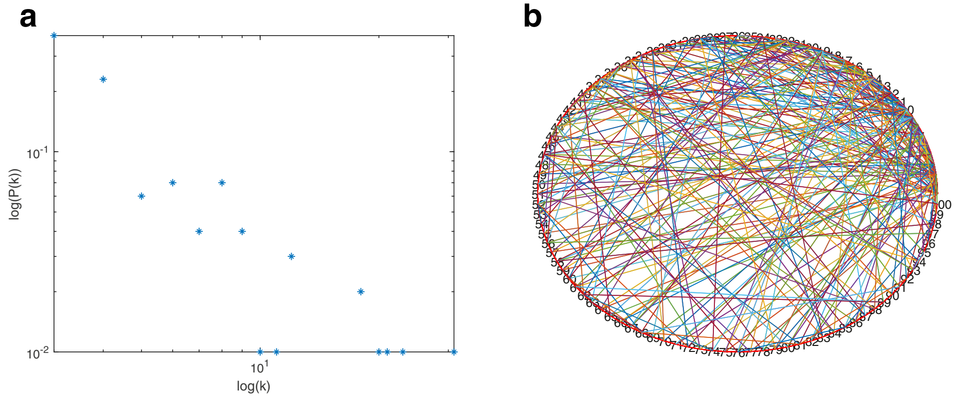

Fig. 5. (a) Evolution of logarithm of node probability $P(k)$ versa logarithm of degree k on a scale-free network with 100 nodes and $m = 3$; (b) a scale-free network with 100 nodes and $m = 3$.

Table 1
The parameters description of the epidemic model.

Parameter	Description	Estimated value	Source
Λ	The constant recruitment rate or birth rate	0.028–0.4 year^{-1}	[24]
γ	The transmission rate from the removed people to the susceptible people	0.05 year^{-1}	Estimate
ε	The transmission rate from the removed people to the infected people	0.6 year^{-1}	Estimate
σ	The effective exposure rate of a susceptible to the infected individuals	0.08 year^{-1}	[23]
μ	The natural death rate	0.01–0.04 year^{-1}	[24]

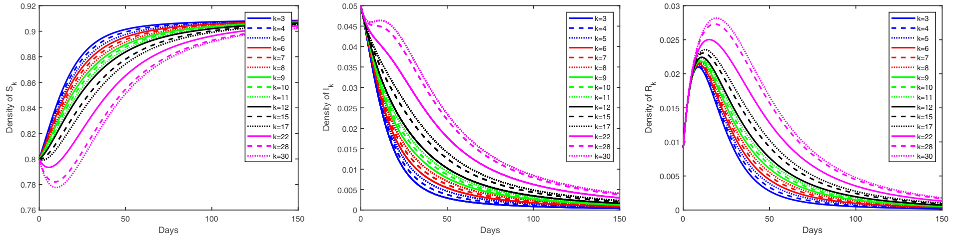

Fig. 6. The globally asymptotical stability of disease-free equilibrium E^0 with different degree if $R_0 < 1$.

Secondly, we choose a set of parameters as following:

$$\Lambda = 0.4\,\text{year}^{-1}, \mu = 0.04\,\text{year}^{-1}, \sigma = 0.08\,\text{year}^{-1}, \varepsilon = 0.6\,\text{year}^{-1}, \gamma = 0.05\,\text{year}^{-1},$$
$$r = 2.4\,\text{year}^{-1}, \beta = 0.2\,\text{year}^{-1}, \alpha = 0.6\,\text{year}^{-1}, \tau = 5\,\text{year}^{-1}, \rho = 0.01\,\text{year}^{-1}.$$

Then $R_0 = 1.7842 > 1$. According to Theorem 3.4.6, the endemic equilibrium E^{**} is globally asymptotically stable, as is shown Fig. 7, where

$$\rho(a) = \begin{cases} 0, & a \le \tau, \\ \rho, & a > \tau, \end{cases} \beta_k(a) = \begin{cases} 0, & a \le \tau, \\ \frac{k\beta}{1+\alpha k}, & a > \tau. \end{cases} \tag{5.1}$$

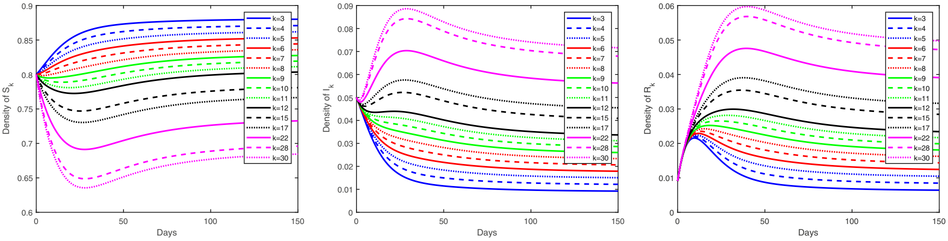

Fig. 7. The globally asymptotical stability of endemic equilibrium E^0 with different degree if $R_0 > 1$.

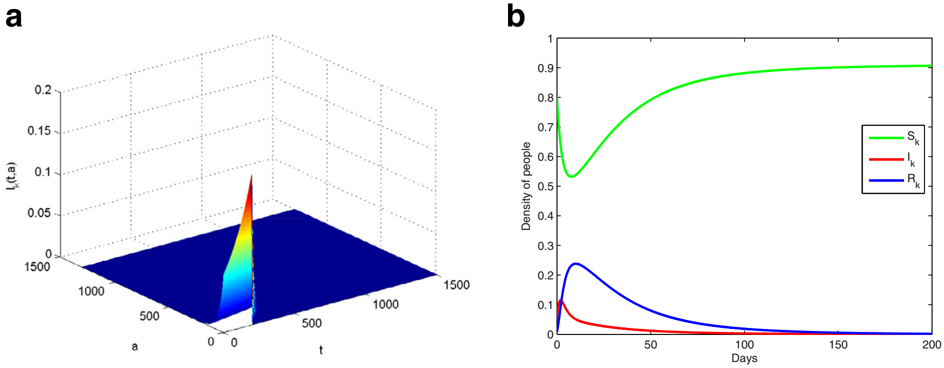

Fig. 8. The globally asymptotical stability of disease-free equilibrium E^0 if $R_0 < 1$. (a) The infected individuals $I_k(t, a)$ with respect to time t and age a. (b)The infected individuals $I_k(t)$ with respect to time.

Here, we assume that the incubation period equals to the cure period. From Figs. 6 and 7, we know that the susceptible individuals decrease as the degree of node increases, however, the infected and removed individuals increase as the degree of nodes increases.

Thirdly, we take the transmission rate and the recovered rate as

$$\rho(a) = \begin{cases} 0, & 0 \le a \le 10, \\ 30(a-10)e^{-0.05(a-25)^2}, & 10 < a \le 40, \\ 0.012, & otherwise, \end{cases} \tag{5.2}$$

and

$$\beta_k(a) = \begin{cases} 0.00002, & 0 \le a \le 10, \\ 0.00002 + 0.00004(a-10)e^{-0.008(a-25)^2}, & 10 < a \le 40, \\ 0.00004, & otherwise. \end{cases} \tag{5.3}$$

If we select a set of parameters as following:

$$\Lambda = 0.028\,\text{year}^{-1}, \mu = 0.04\,\text{year}^{-1}, \sigma = 0.08\,\text{year}^{-1}, \varepsilon = 0.6\,\text{year}^{-1}, \gamma = 0.05\,\text{year}^{-1},$$
$$r = 2.4\,\text{year}^{-1}, \beta = 0.2\,\text{year}^{-1}, \alpha = 0.6\,\text{year}^{-1}, \tau = 5\,\text{year}^{-1}, \rho = 0.01\,\text{year}^{-1}.$$

Then $R_0 = 0.6472 < 1$ and the disease-free equilibrium E^0 is also globally asymptotically stable, see Fig. 8. If we select $\Lambda = 0.28$, then $R_0 = 1.5683 < 1$ and the endemic equilibrium E^{**} is also globally asymptotically stable, see Fig. 9. From Figs. 8 and 9, we obtain that the number of infected individuals reduces almost to 0, and it also increases quickly to a peak.

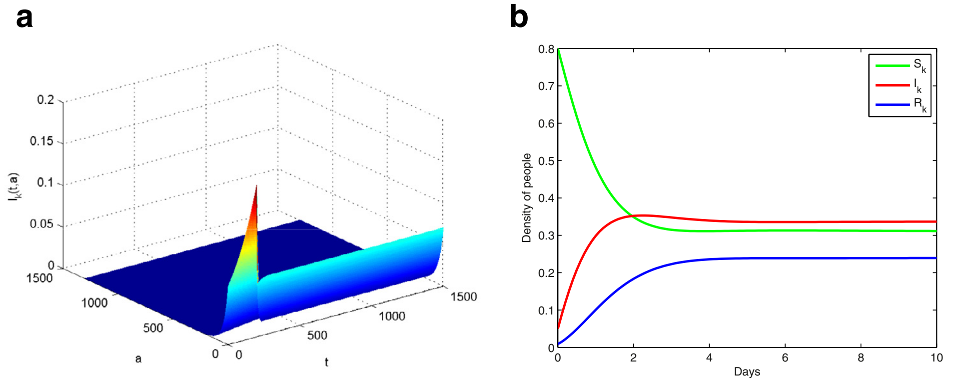

Fig. 9. The globally asymptotical stability of endemic equilibrium E^{**} if $R_0 > 1$. (a) The infected individuals $I_k(t, a)$ with respect to time t and age a. (b) The infected individuals $I_k(t)$ with respect to time.

Therefore, the damped occupation phenomenon occurs, and it indicates that infection age contribute to multiple peaks of infection, which strengthen the difficulty of disease control. Comparing Figs. 6 and 7 with Figs. 8 and 9, we have different infection age distribution function and degree of nodes result in different transmission trends for a short time infectious period. Thus, infection age and degree of nodes play an important role on the disease spread in initial transmission period.

6. Sensitivity analysis and discussion

According to Theorems 3.2.2 and 3.4.6, we know the dynamical behavior of system (2.11) is determined by the basic reproduction number R_0. Hence, we perform the sensitivity analysis of R_0.

We set that the form of $\rho(a)$ and $\beta_k(a)$ is similar to that of (5.1). Therefore,

$$R_0 = \frac{\sigma\left\langle \frac{(k-1)k^2}{1+\alpha k}\beta e^{-\mu\tau} S_j^0\right\rangle}{\langle k\rangle(\mu+\rho)}. \tag{6.1}$$

Therefore,

$$\frac{\partial R_0}{\partial \sigma} = \frac{\left\langle \frac{(k-1)k^2}{1+\alpha k}\beta e^{-\mu\tau} S_j^0\right\rangle}{\langle k\rangle(\mu+\rho)} > 0,\ \frac{\partial R_0}{\partial \beta} = \frac{\left\langle \frac{(k-1)k^2}{1+\alpha k} e^{-\mu\tau} S_j^0\right\rangle}{\langle k\rangle(\mu+\rho)} > 0, \tag{6.2}$$

$$\frac{\partial R_0}{\partial \rho} = -\frac{\left\langle \frac{(k-1)k^2}{1+\alpha k}\beta e^{-\mu\tau} S_j^0\right\rangle}{\langle k\rangle(\mu+\rho)^2} < 0,\ \frac{\partial R_0}{\partial \tau} = -\frac{\mu\left\langle \frac{(k-1)k^2}{1+\alpha k}\beta e^{-\mu\tau} S_j^0\right\rangle}{\langle k\rangle(\mu+\rho)^2} < 0. \tag{6.3}$$

We take

$$\Lambda = 0.028\,\text{year}^{-1},\ \mu = 0.01\,\text{year}^{-1},\ r = 2.4\,\text{year}^{-1},\ \rho = 0.01\,\text{year}^{-1},\ \tau = 5\,\text{year}^{-1},$$
$$\sigma = 0.08\,\text{year}^{-1}. \tag{6.4}$$

From Fig. 10, we obtain that the basic reproduction number R_0 increases when the transmission rate β and exposed rate σ increase, however, the basic reproduction number R_0 decreases

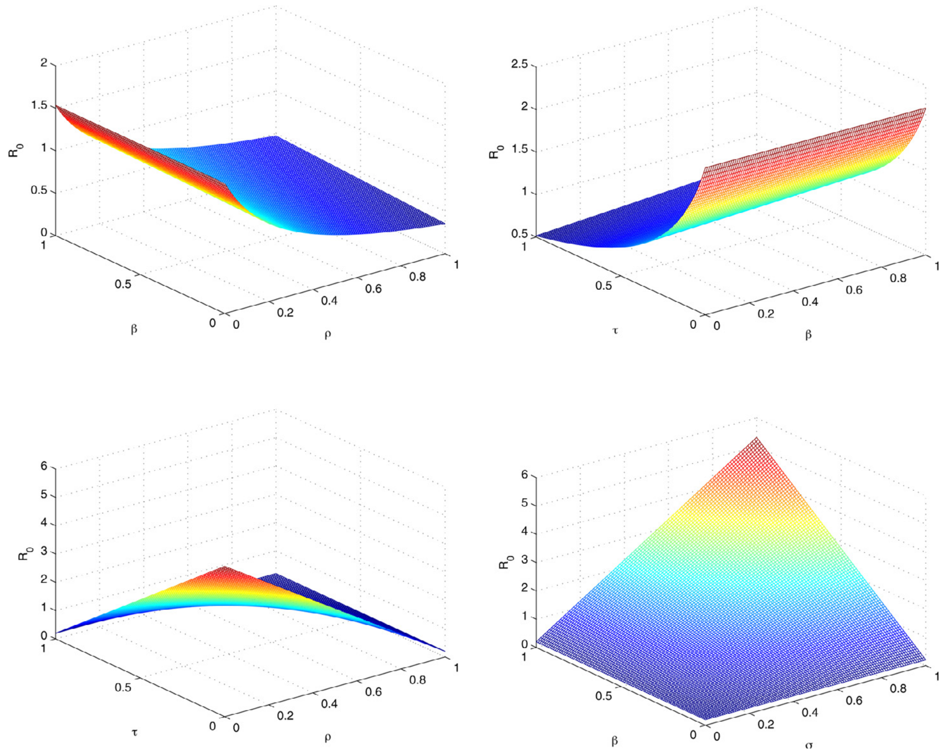

Fig. 10. The relationship between the basic reproduction number R_0 and the relevant parameters.

when the removed rate ρ and τ increase. Therefore, decreasing the transmission rate β or exposed rate σ and increasing the removed rate ρ or τ are effective methods to control the spread of epidemic disease.

From Fig. 11, we know that the infected number $I_k(t)$ increases when the relapse rate ε and the exposed rate σ increase, though, the infected number $I_k(t)$ decreases when the removed rate ρ and τ increase. Thus, the relapse rate ε has an important effect to the spread of epidemic disease. We know that preventing relapse and the temporary immunization of the removed individuals are very significant in transmission process of epidemic disease.

In this paper, we consider the effect of the infection age and a scale-free network to epidemic disease, meanwhile, we take the relapse rate and the temporary immunity of the removed individuals into consideration to epidemic disease. Hence, we construct an SIRS epidemic model with infection age and relapse on a scale-free network, then we obtain that the disease will die out if the basic reproduction number is less than one, otherwise, the disease outbreaks. Furthermore, we propose some efficient control measures to prevent the spread of disease on a scale-free network. Deducing number of high risk susceptible individuals and avoiding the relapse of removed people are effective measures to control the spread of epidemic disease. It is a challenging work to study the effect of topological structure of networks on the transmission process of epidemic model, whether there are forward, backward, Hopf bifurcations or other complicated dynamical behaviours can be explored. On the other hand, if we consider the physiological age of an individual, it is enough to classify the nodes

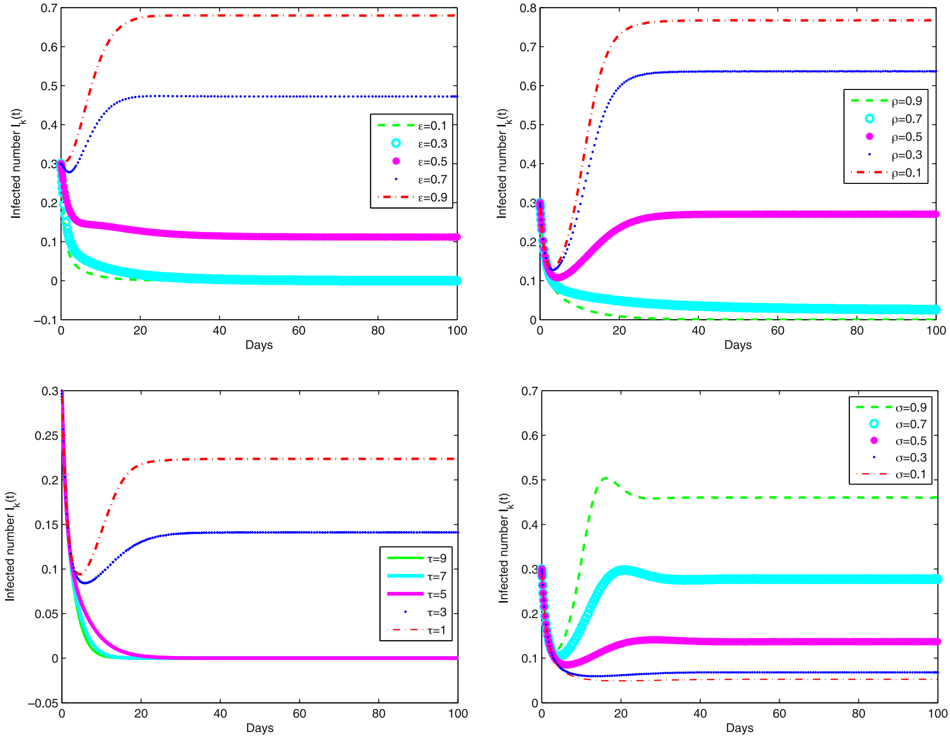

Fig. 11. The relationship between the infected number $I_k(t)$ and the relevant parameters.

into several age-groups, such as, child, adult and old people. If we study a discrete version of our model, age can be considered as the discrete variance rather than the continuous case, in this situation, the summation can replace the integral. We leave these interesting works for the future.

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