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Fujita type critical exponent for a free boundary problem with spatial–temporal source

Jie Wang *, Jia-Feng Cao

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, People's Republic of China
State Key Laboratory of Advanced Processing and Recycling of Non-ferrous Metals, Lanzhou, Gansu 730050, People's Republic of China

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A B S T R A C T

In this paper, we investigate a nonlinear free boundary problem incorporating with nontrivial spatial and exponential temporal weighted source. To portray the asymptotic behavior of the solution, we first derive some sufficient conditions for finite time blowup. Furthermore, the global vanishing solution is also obtained for a class of small initial data. Finally, a sharp threshold trichotomy result is provided in terms of the size of the initial data to distinguish the blowup solution, the global vanishing solution, and the global transition solution. In particular, our results show that such a problem always possesses a Fujita type critical exponent whenever the spatial source is just equivalent to a trivial constant, or is an extreme one, such as "very negative" one in the sense of measure or integral.

1. Introduction

As is well-known, a number of diffusion and growth phenomena in natural world, which varies from temperature change in chemical reaction to population invasion and expanding, can be modeled by the following nonlinear reaction–diffusion equation:

$$u_t(t,x) = d\Delta u + f(t,x,u(t,x)), \quad t > 0, x \in \mathbb{R}^N. \tag{1.1}$$

Here, the diffusion term Δu describes the spatial migration behavior of the temperature, population density, and so on, while the reaction term $f(t,x,u(t,x))$ represents the resource supporting their growth. Then Eq. (1.1) indicates that the two factors together lead to the change of state variable u (see [1]).

In various applied fields, if the interaction between the diffusivity and growth is complex enough or the growth is extensively nonlinear, then many interesting phenomena arise. For example, the chemical reaction

* Corresponding author at: Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, People's Republic of China

E-mail addresses: jiema138@163.com (J. Wang), caojf07@lzu.edu.cn (J.-F. Cao).

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with a high initial temperature will generate lots of heat resulting in the higher temperature, while the higher temperature will accelerate the chemical reaction again, which implies that the temperature will become as high as it can be, and even converge to infinity in finite time. However, the reaction temperature will grow slowly in a long period when the initial temperature is small enough. Similar phenomena are also observed in population dynamics, in which the large initial population density will cause a potential population explosion, while the species may experience a long term slow growth for a small initial population density. Mathematically, the former case means the solution blows up, while the later one implies the global existence arises.

Historically, Fujita [2] initiated the study on the key conception "critical exponent" to explore such problems. More precisely, [2] and its aftermath [3,4] revealed that the Cauchy problem of the following semilinear heat equation

$$u_t = d\Delta u + u^p, \quad t > 0, \quad x \in \mathbb{R}^n \tag{1.2}$$

admits no nontrivial and nonnegative global solution for any $1 < p \le P_c := 1 + 2/n$; however, both local and global solutions exist provided $p > P_c$, depending on the size of initial data. Thereafter, a large number of similar results were established by many researchers to various kinds of evolution problems, and one can refer to the surveys [5,6] and monograph [7] and the references cited therein.

Generally speaking, the critical exponent is just considered for unbounded domains since the global solutions always exist in bounded domains. However, it was found by Meier [8] that for the following problem

$$u_t = \Delta u + e^{\beta t} u^p, \quad t > 0, \quad x \in \Omega \tag{1.3}$$

defined in a bounded domain Ω with temporal weighted source, the critical Fujita exponent still exists with the form

$$P_c = 1 + \frac{\beta}{\lambda_1}, \tag{1.4}$$

where $\beta > 0$ is an exponent and λ_1 is the first Dirichlet eigenvalue of the Laplacian operator in Ω. Since that, many papers are devoted to investigating such class of problems, and the similar Fujita type critical exponents were derived, see e.g., [9–11] for some parabolic systems and [12] for nonlocal diffusion equations.

On the other hand, the free boundary problems have attracted much attention in recent years. In particular, Fila and Souplet [13] and Ghidouche et al. [14] first studied the blowup and global existence of a similar one-dimensional problem to (1.2) with one free boundary, and in their work, the global solutions were distinguished to the fast solution and the slow one, respectively. After that, similar results were obtained by Zhou et al. [15], Zhou and Lin [16] and Yang [17] for heat equations with local or nonlocal nonlinear reactions, and [18] for a semilinear cooperative system. Very recently, Sun [19] obtained a complete description on the long-time dynamical behavior of the solutions to one type of reaction–diffusion equations, in which the additional spatial source a in the problem (1.5) is a constant and varies from positive, negative to 0. Meanwhile, [19] also presents a sharp threshold trichotomy result, by which the blowup solution, the global vanishing solution, and the global transition solution are distinguished. Regarding the free boundary problems, we refer the readers to Crank's monograph [20] for further understanding.

Inspired by the works mentioned above, the main purpose of the current paper is to study the spatial dynamics of the following free boundary problem with nontrivial spatio-temporal source:

$$\begin{cases} u_t = du_{xx} + a(x)u + e^{\beta t}u^p, & t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) = 0,\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0,\ h'(t) = -\mu u_x(t, h(t)),\ t > 0, & \\ -\ g(0) = h_0 = h(0),\ u(0, x) = u_0(x), -h_0 \le x \le h_0, & \end{cases} \tag{1.5}$$

where $a \in (C^1 \cap L^\infty)((-\infty, \infty))$ is the spatial source, the exponent $p > 1$ represents the superlinearity, both $x = g(t)$ and $x = h(t)$ are free boundaries (expanding fronts), $\mu > 0$ denotes the expanding capability of free boundaries, $h_0 > 0$ is the size of initial boundary and $d > 0$ is the diffusion coefficient. Throughout this paper, we always assume that the initial function u_0 belongs to $\mathscr{X}(h_0)$ for fixed h_0, where

$$\mathscr{X}(h_0) := \{u_0 \in C^2([-h_0, h_0]) : u_0(-h_0) = u_0(h_0) = 0 \text{ and } u_0 > 0 \text{ in } (-h_0, h_0)\}. \tag{1.6}$$

Based on many former studies where the spatial source a is assumed to be a positive or negative constant [17,19], or 0 directly [18], in this paper we will show that a Fujita type critical exponent P_c^0 still exists whenever the trivial case that spatial source function $a(x)$ is just equivalent to a constant $a > 0$ or the extreme case that $a(x)$ is, for example, "very negative" in the sense of measure or integral (see [21,22]). To our best knowledge, the present paper seems to be the first attempt to investigate the critical exponent for the nonlinear reaction–diffusion problems with free boundaries.

Finally, we introduce several concepts which will be used later. Define the maximal existence time as follows:

$$T_{\max} = T_{\max}(u_0) := \sup\{T > 0 : \text{ classical solution exists on } [0, T) \text{ for } u_0\}.$$

When $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = \infty$, we say that the solution u blows up in finite time and $T_{\max}$ is thus known as the blowup time, otherwise the solution exists globally. Indeed, it follows from Theorem 3.4 that the solution must blow up in finite time as long as $T_{\max} < \infty$. Moreover, both the global vanishing solution and global transition solution of problem (1.5) exist in the sense that $T_{\max} = \infty$, solution $u(t, x)$ decays uniformly to zero, and the free boundaries ultimately constitute a finite interval with the length either less than or equivalent exactly to a certain critical value.

The rest of this paper is arranged as follows. Section 2 will be contributed to state some preliminaries regarding the existence, uniqueness, regularity and estimate of the solutions, and make proper assumptions on the spatial source function $a(x)$. In Section 3, we are concerned with the blowup property. By constructing an auxiliary functional and applying the comparison principle, we establish some sufficient conditions to finite time blowup (see Theorem 3.1). Then Section 4 is devoted to the global existence of solutions with sufficiently small initial data (Theorem 4.1). The proof relies on the technical construction of a suitable global upper solution, which is new in the literature. Thus, we identify the Fujita type critical exponent of (1.5) determined by the spatio-temporal heterogeneity. In Section 5, by employing the indirect argument and the comparison principle, we prove a sharp threshold trichotomy result in terms of the size of the initial data, by which the blowup solution, the global vanishing solution, and, in particular, the global transition solution are distinguished, see Theorem 5.1. Finally, some discussion about our future research will be given in Section 6.

2. Preliminaries

2.1. Existence, uniqueness, regularity and estimate of the solution

In this subsection, for completeness, we give the well-posedness, boundness and monotonicity of the solution to (1.5) as well as the comparison principle. Note that the proof methods of them are very standard (such as the contraction mapping theorem, the Hopf lemma and the maximum principle) and we refer the readers to a similar discussion in [16, Section 2] or [17,23–25] for more details.

Theorem 2.1. *Assume $p > 1$ and $a \in (C^1 \cap L^\infty)((-\infty, \infty))$. For any given $u_0 \in \mathscr{X}(h_0)$ and $\alpha \in (0, 1)$, there is a constant $T > 0$ such that the problem (1.5) admits a unique positive solution*

$$(u, g, h) \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_T) \times C^{1+\frac{\alpha}{2}}([0, T]) \times C^{1+\frac{\alpha}{2}}([0, T]);$$

moreover,

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\overline{D}_T)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} < C, \tag{2.1}$$

where

$$D_T = \{(t,x) \in \mathbb{R}^2 : t \in [0,T], x \in [g(t), h(t)]\},$$

and the constants T *and* C *only depend on* h_0, α, *and* $\|u_0\|_{C^2([-h_0,h_0])}$.

Lemma 2.2. *Assume* $p > 1$ *and* $a \in (C^1 \cap L^\infty)((-\infty,\infty))$. *Let* (u,g,h) *be the solution of the problem (1.5) defined for* $t \in [0,T_0)$ *for some* $T_0 \in$, *and there exists* $M_1(T_0)$ *such that* $u(t,x) \leq M_1$ *for* $0 \leq t < T_0$ *and* $g(t) \leq x \leq h(t)$. *Then there exists a constant* $C(T_0) > 0$ *such that*

$$0 < -g'(t), \quad h'(t) \leq C(T_0) \tag{2.2}$$

and

$$h_0 < -g(t), \quad h(t) \leq h_0 + C(T_0)T_0 \tag{2.3}$$

for $0 \leq t < T_0$.

Lemma 2.3. *Assume* $p > 1$ *and* $a \in (C^1 \cap L^\infty)((-\infty,\infty))$. *Let* $T \in (0, T_{\max})$, $\overline{g}, \overline{h} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}_T^*) \cap C^{1,2}(D_T^*)$ *with*

$$D_T^* := \{(t,x) \in \mathbb{R}^2 : \ t \in (0,T], \ x \in (\overline{g}(t), \overline{h}(t))\},$$

and

$$\begin{cases} \overline{u}_t - d\overline{u}_{xx} \geq a(x)\overline{u} + e^{\beta t}\overline{u}^p, & t > 0, \ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u}(t,\overline{g}(t)) = 0, \ \overline{g}'(t) \leq -\mu\overline{u}_x(t,\overline{g}(t)), \ t > 0, \\ \overline{u}(t,\overline{h}(t)) = 0, \ \overline{h}'(t) \geq -\mu\overline{u}_x(t,\overline{h}(t)), \ t > 0. \end{cases}$$

If (u,g,h) *is the solution of the problem (1.5), and satisfies*

$$\overline{g}(0) \leq -h_0, \ \overline{h}(0) \geq h_0 \text{ and } \overline{u}(0,x) \geq u_0(x) \text{ in } [-h_0,h_0],$$

it follows that

$$g(t) \geq \overline{g}(t), \ h(t) \leq \overline{h}(t) \text{ in } (0,T],$$
$$u(t,x) \leq \overline{u}(t,x) \text{ for } (t,x) \in (0,T] \times (g(t),h(t)).$$

Remark 2.4. Henceforth, the triple $(\overline{u}, \overline{g}, \overline{h})$ or the function $\overline{u}$ alone sometime, in Lemma 2.3, is often called an upper solution of (1.5). A lower solution can be defined analogously by reversing all the inequalities above.

2.2. Assumptions on the spatial source

To further investigate the effect of the nontrivially spatial source function $a(x)$ on the asymptotic behavior of the solution, we first consider the following eigenvalue problem

$$\begin{cases} d\psi_{xx} + a(x)\psi + \lambda\psi = 0, & -L < x < L, \\ \psi = 0, & x = \pm L. \end{cases} \tag{2.4}$$

For any fixed $d > 0$ and $a(x)$, denote the principal eigenvalue of (2.4) by $\lambda_1 = \lambda_1(L;a)$. Now, if we suppose that spatial source function $a(x)$ has the property that

(H) $\lim_{L\to\infty} \lambda_1(L;a) \leq 0$,

then by [21, Proposition 3.1] and the remark following it, we have the following conclusion.

Proposition 2.5. *For any fixed $d > 0$ and $a(x)$, the principal eigenvalue of (2.4), denoted by $\lambda_1 = \lambda_1(L; a)$, is continuous and strictly decreasing in $L > 0$, and $\lim_{L\to 0^+} \lambda_1 = +\infty$. Moreover, if $a(x)$ satisfies **(H)**, then for any $c \in (0, \infty)$, there exists a unique $L_c > 0$ such that $\lambda_1 = c$ if and only if $L = L_c$.*

We should point out that the assumption **(H)** contains many common cases for $a(x)$. For example, it is clear that **(H)** holds provided that $a(x) \equiv a > 0$, since we have $\lambda_1(L; a) = \frac{d\pi^2}{4L^2} - a$ in such case. At the same time, some more complex cases are still possible. For instance, assume that $a(x)$ satisfies

(A) There exist $\eta > 0$, $-2 < \rho \le 0$, $k > 1$ and x_n satisfying $x_n \to \infty$ as $n \to \infty$, such that $a(x) \ge \eta|x|^\rho$ in $[x_n, kx_n]$,

it then follows from [21, Proposition 3.2] or [26, Proposition 2.6] that assumption **(H)** holds. Differing significantly from the former case, it seems that $a(x)$ is "*very negative*" under the condition **(A)**, since both the case that $|\{a(x) > 0\}| \ll |\{a(x) < 0\}|$ and the case that $\int_0^\infty a(x)\mathrm{d}x = -\infty$ are possible. Henceforth, we always assume that condition **(H)** holds to the spatial source function $a(x)$, and denote the "*0-critical length*" by L_0, for which $\lambda_1(L_0; a) = 0$. Here, we naturally set $L_0 = \infty$ provided that $\lim_{L\to\infty} \lambda_1(L; a) = 0$.

3. Finite time blowup

In this section, we will derive some sufficient conditions for finite time blowup. Let $L = h_0$ in the eigenvalue problem (2.4) and denote by λ_1^0 the corresponding principal eigenvalue with a positive eigenfunction $\psi_1^0 > 0$ in $(-h_0, h_0)$, which is suitably standardized by $\|\psi_1^0\|_{L^1([-h_0,h_0])} = 1$. Moreover, we define

$$P_c^0 := P_c(h_0) = 1 + \frac{\beta}{\lambda_1^0} \tag{3.1}$$

for $\lambda_1^0 > 0$ and set $P_c^0 = +\infty$ for $\lambda_1^0 = 0$, respectively. The following theorem shows that P_c^0 is partly the critical exponent we wanted.

Theorem 3.1. *Assume $p > 1$, $\beta > 0$, $h_0 \in (0, L_0]$, $\phi \in \mathscr{X}(h_0)$ and **(H)** holds. Then the solution of the problem (1.5) with $u_0(x) = \delta\phi(x)$ will blow up in finite time if one of the following conditions is valid:*

(i) *$1 < p \le P_c^0$ and $\delta > 0$;*

(ii) *$p > P_c^0$ and $\delta > \delta_* := \left[\frac{\lambda_1^0(p-1)-\beta}{p-1}\right]^{\frac{1}{p-1}} E_0^{-1}$, where $E_0 := \int_{-h_0}^{h_0} \phi\psi_1^0 \mathrm{d}x$.*

Further, $T_{\max} \le \widetilde{C}\delta^{-(p-1)}$, where the constant $\widetilde{C} > 0$ depends on $a(x)$, p, β, h_0 and ϕ.

Proof. We first introduce the following auxiliary problem:

$$\begin{cases} v_t = dv_{xx} + a(x)v + e^{\beta t}v^p, & 0 < t < \widetilde{T}_{\max}, \quad -h_0 < x < h_0, \\ v(t, -h_0) = v(t, h_0) = 0, & 0 < t < \widetilde{T}_{\max}, \\ v(0, x) = \delta\phi(x), & -h_0 \le x \le h_0, \end{cases} \tag{3.2}$$

where $\widetilde{T}_{\max}$ is the maximal time of $v(t, x)$. Note that the comparison principle yields that $T_{\max} \le \widetilde{T}_{\max}$ and $u(t, x) \ge v(t, x)$ on $[0, T_{\max}) \times [-h_0, h_0]$. Thus, to show that $u(t, x)$ blows up in finite time, it is enough to prove that the subsolution $v(t, x)$ blowups in finite time. Motivated from [27], we construct an auxiliary functional as follows:

$$E(t) := \int_{-h_0}^{h_0} v(t, x)\psi_1^0(x)\mathrm{d}x.$$

Then, by virtue of the Green's identity and Jensen's integral inequality, we obtain that

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\int_{-h_0}^{h_0}v_t\psi_1^0\mathrm{d}x=\int_{-h_0}^{h_0}\left[dv_{xx}+a(x)v\right]\psi_1^0\mathrm{d}x+\int_{-h_0}^{h_0}e^{\beta t}v^p\psi_1^0\mathrm{d}x\geq-\lambda_1^0E+e^{\beta t}E^p.$$

Using the comparison principle of the linear ODE, we have

$$E^{1-p}\leq\left\{\delta^{1-p}E_0^{1-p}-\frac{1-p}{\beta-\lambda_1^0(p-1)}\left[1-e^{[\beta-\lambda_1^0(p-1)]t}\right]\right\}e^{\lambda_1^0(p-1)t},$$

which implies that

$$E^{p-1}\geq\frac{1}{\left\{\delta^{1-p}E_0^{1-p}-\frac{1-p}{\beta-\lambda_1^0(p-1)}\left[1-e^{[\beta-\lambda_1^0(p-1)]t}\right]\right\}e^{\lambda_1^0(p-1)t}}. \tag{3.3}$$

Therefore, we are leading to the following two cases:

(i) If $1<p<P_c^0$, then we have $\beta-\lambda_1^0(p-1)>0$ and $\frac{1-p}{\beta-\lambda_1^0(p-1)}<0$. Note that $\delta^{1-p}E_0^{1-p}>0$ for all $\delta>0$, it then follows from (3.3) that $v(t,x)$ blows up in finite time for any $\delta>0$. Similarly, in the case where $p=P_c^0<+\infty$, L'Hopital's Rule together with the inequality (3.3) yields that

$$E^{p-1}\geq\frac{1}{\left[\delta^{1-p}E_0^{1-p}-(p-1)t\right]e^{\lambda_1^0(p-1)t}}, \tag{3.4}$$

which implies that the blowup occurs again.

(ii) If $p>P_c^0$, it follows that

$$\beta-\lambda_1^0(p-1)<0\text{ and }\frac{1-p}{\beta-\lambda_1^0(p-1)}>0.$$

Therefore, when

$$\delta^{1-p}E_0^{1-p}-\frac{1-p}{\beta-\lambda_1^0(p-1)}<0,$$

that is,

$$\delta>\delta_*:=\left[\frac{\lambda_1^0(p-1)-\beta}{p-1}\right]^{\frac{1}{p-1}}E_0^{-1},$$

using (3.3) again, we see that $v(t,x)$ blows up in finite time.

Moreover, since whatever (3.3) and (3.4) can imply that $\widetilde{T}_{\max}\leq\widetilde{C}\delta^{-(p-1)}$ with constant $\widetilde{C}>0$ dependent on $a(x)$, p, β, h_0 and ϕ, the comparison principle implies that the same estimate holds for $T_{\max}$. The proof is now completed. □

Remark 3.2. In view of Theorem 3.1, if we define

$$\Lambda:=\Lambda(h_0,\phi)=\begin{cases}0, & \text{if } 1<p\leq P_c^0,\\ \delta_*, & \text{if } p>P_c^0.\end{cases} \tag{3.5}$$

then the solution of the problem (1.5) with initial data $u_0=\delta\phi$ will blow up when $\delta>\Lambda$.

By Proposition 2.5, since $1<p\leq P_c^0$ implies $0\leq\lambda_1^0\leq\frac{\beta}{p-1}$, it is clear that the later is equivalent to $L_c^*\leq h_0\leq L_0$, where L_c^* is determined by Proposition 2.5 with $c=c^*:=\frac{\beta}{p-1}$. Thus, we immediately obtain the following result.

Corollary 3.3. *Assume $p>1$, $\beta>0$ and **(H)** holds. If $L_{c^*}\leq h_0\leq L_0$, then every solution of the problem (1.5) blows up in finite time.*

By the identical argument to [16, Theorem 3.2], one can prove the following continuation theorem, which implies that the solution of problem (1.5) blows up provided that the maximal existence time $T_{\max}$ is finite.

Theorem 3.4. *Let (u, g, h) be the solution of the problem (1.5) with the maximal existence time $T_{\max}$. If $T_{\max} < \infty$, then*

$$\lim_{t \to T_{\max}} \|u(t, \cdot)\|_{L^\infty([g(t), h(t)])} = \infty. \tag{3.6}$$

4. Global existence

In this section, we consider the existence of global solution of (1.5). Based on Lemma 2.2, we can define $g_\infty := \lim_{t\to\infty} g(t)$ and $h_\infty := \lim_{t\to\infty} h(t)$. The following theorem shows that the global vanishing solution of (1.5) exists for the sufficiently small initial value, for which $T_{\max} = \infty$, solution $u(t, x)$ decays uniformly to zero, and $h_\infty - g_\infty < 2L_{c^*}$.

Theorem 4.1. *Assume $p > P_c^0$, $\beta > 0$, $h_0 \in (0, L_0)$ and **(H)** hold. If the initial value $\|u_0\|_{L^\infty([-h_0, h_0])}$ is sufficiently small, then $T_{\max} = \infty$ and (g_∞, h_∞) is a finite interval with length less than $2L_{c^*}$. Moreover, there exist some real numbers C and $\nu > 0$ depending on u_0 and h_0 such that*

$$\|u(t, \cdot)\|_{L^\infty([g(t), h(t)])} \le Ce^{-\nu t}, \quad \forall t > 0. \tag{4.1}$$

Proof. Inspired by [14], the essential idea of the proof is to construct a suitable global upper solution. Let the eigenfunction ψ_1^0 associated with λ_1^0 satisfy $\psi_1^0 > 0$ in $(-h_0, h_0)$ and $\psi_1^0(-h_0) = \psi_1^0(h_0) = 0$. We first note that $\frac{\mathrm{d}\psi_1^0(h_0)}{\mathrm{d}x} < 0$. Furthermore, the regularity of ψ_1^0 implies that there exists a constant $K > 0$ such that

$$x\frac{\mathrm{d}\psi_1^0(x)}{\mathrm{d}x} \le K\psi_1^0(x), \quad \forall x \in [-h_0, h_0]. \tag{4.2}$$

Define

$$s(t) := h_0\big(1 + 2\sigma - \sigma e^{-\eta t}\big) \ \text{ and } \ v(t, x) := z(t)\|\psi_1^0\|_\infty^{-1} e^{\left(-\frac{\beta}{p-1} - \eta\right)t} \psi_1^0\left(\frac{xh_0}{s(t)}\right),$$

where $0 < \sigma, \eta < 1$ are constants which will be determined later, $z(t)$ is the positive solution of the following ODE

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = e^{-\frac{p-1}{p}\eta t} z^p(t), \\ z(0) = z_0, \end{cases}$$

in which $0 < z_0 \le \left(\frac{\eta}{p}\right)^{\frac{1}{p-1}}$, and $\|\psi_1^0\|_\infty := \|\psi_1^0\|_{L^\infty([-h_0, h_0])} + 1$. By a direct calculation, we obtain that

$$z(t) = \left\{ z_0^{1-p} - \frac{p}{\eta}\left[1 - e^{-\frac{p-1}{p}\eta t}\right] \right\}^{\frac{1}{1-p}}.$$

Clearly, $v(t, -s(t)) = v(t, s(t)) = 0$. Since $p > P_c^0$, it follows that $\lambda_1^0 > \frac{\beta}{p-1}$, which implies $0 < h_0 < L_{c^*}$. We thus first choose $0 < \sigma < \frac{1}{2}\big(\frac{L_{c^*}}{h_0} - 1\big)$ such that

$$h_0(1 + \sigma) \le s(t) < h_0(1 + 2\sigma) < L_{c^*}, \quad \forall t > 0.$$

Moreover, for any given $0 < \epsilon \ll 1$, since $\lambda_1^0 > \frac{\beta}{p-1}$, the continuous dependence of $s(t)$ on σ and the uniform boundedness of $a(x)$ in $[-3h_0, 3h_0]$ imply that there exists $0 < \sigma_0(\epsilon) \ll 1$ such that, for any $0 < \sigma \le \sigma_0(\epsilon)$ and $0 < \eta < 1$, there hold that

$$\left|\left[\frac{h_0^2}{s^2(t)} - 1\right] a(x)\right| \le \frac{\epsilon}{2}, \quad \forall t > 0, x \in [-s(t), s(t)],$$

and

$$\left|\lambda_1^0 \frac{1}{(1+2\sigma)^2} - \frac{\beta}{p-1}\right| \le \frac{\epsilon}{2}.$$

Considering the facts above together with (4.2), it follows that

$$\begin{aligned}
& v_t - dv_{xx} - a(x)v - e^{\beta t} v^p \\
& \ge \left\{ -\frac{\beta}{p-1} - \eta - \frac{\mathrm{d}\psi_1^0}{\mathrm{d}x}\left(\frac{xh_0}{s(t)}\right)\left[\psi_1^0\left(\frac{xh_0}{s(t)}\right)\right]^{-1} \frac{xh_0\sigma\eta}{s^2(t)} e^{-\eta t} + \left[\frac{h_0^2}{s^2(t)} - 1\right] a(x) + \lambda_1^0 \frac{h_0^2}{s^2(t)} \right\} v \\
& + \left\{ z^p(t) e^{-\frac{p-1}{p}\eta t} e^{\left(-\frac{\beta}{p-1} - \eta\right)t} \|\psi_1^0\|_\infty^{-1} \psi_1^0\left(\frac{xh_0}{s(t)}\right) - z^p(t) e^{\beta t} e^{\left(-\frac{\beta}{p-1}-\eta\right)pt} \left[\|\psi_1^0\|_\infty^{-1} \psi_1^0\left(\frac{xh_0}{s(t)}\right)\right]^p \right\} \\
& \ge \left[-\epsilon - (K+1)\eta\right] v \\
& + z^p(t) e^{\left(-\frac{\beta}{p-1}-\eta\right)t} \left\{ e^{-\frac{p-1}{p}\eta t} \|\psi_1^0\|_\infty^{-1} \psi_1^0\left(\frac{xh_0}{s(t)}\right) - e^{-\eta(p-1)t} \left[\|\psi_1^0\|_\infty^{-1} \psi_1^0\left(\frac{xh_0}{s(t)}\right)\right]^p \right\} \\
& \ge 0, \quad \forall t > 0,\ x \in [-s(t), s(t)]
\end{aligned}$$

provided that the small parameters ϵ and η are chosen properly. On the other hand, by the strict inequality $\frac{\mathrm{d}\psi_1^0(h_0)}{\mathrm{d}x} < 0$, for fixed $\eta > 0$ and $\sigma > 0$, we can further choose a smaller $z_0 > 0$ such that

$$\begin{aligned}
-\mu v_x(t, s(t)) &= -\frac{\mu h_0}{s(t)} z(t) e^{(-\frac{\beta}{p-1}-\eta)t} \|\psi_1^0\|_\infty^{-1} \frac{\mathrm{d}\psi_1^0(h_0)}{\mathrm{d}x} \\
&\le -\mu\big(z_0^{1-p} - \frac{p}{\eta}\big)^{\frac{1}{1-p}} \|\psi_1^0\|_\infty^{-1} \frac{\mathrm{d}\psi_1^0(h_0)}{\mathrm{d}x} e^{-\eta t} \\
&\le \sigma\eta h_0 e^{-\eta t} \\
&= s'(t).
\end{aligned}$$

Similarly, we also have $-\mu v_x(t, -s(t)) \ge -s'(t)$. Therefore, if $u_0(x) \le v(0, x)$ in $[-h_0, h_0]$, then the comparison principle (Lemma 2.3) implies that $(v(t, x), -s(t), s(t))$ is an upper solution of (1.5). In fact, if we choose

$$\delta^* := z_0 \|\psi_1^0\|_\infty^{-1} \min_{-\frac{h_0}{1+\sigma} \le x \le \frac{h_0}{1+\sigma}} \psi_1^0(x),$$

then when $\|u_0\|_{L^\infty([-h_0,h_0])} \le \delta^*$, we have $u_0(x) \le \delta^* \le v(0, x)$ in $[-h_0, h_0]$. Again, Lemma 2.3 yields that

$$-h_0(1+2\sigma) < -s(t) \le g(t) \text{ and } h(t) \le s(t) < h_0(1+2\sigma), \quad \forall t > 0$$

and

$$u(t, x) \le v(t, x) \le Ce^{-\nu t}, \quad \forall t \ge 0, x \in [-s(t), s(t)],$$

where real numbers C and $\nu > 0$ depend on u_0 and h_0. Thus, Theorem 3.4 implies that $T_{\max} = \infty$. The proof is then finished. □

Remark 4.2. In view of Theorems 3.1 and 4.1, we see that the problem (1.5) possesses the **Fujita critical exponent** $P_c^0 := 1 + \frac{\beta}{\lambda_1^0}$.

The following lemma provides a comprehensive description on the global solutions of (1.5), which reveals that all of the global solutions are bounded and decay uniformly to zero for the case that the spatial source $a(x)$ is trivial nonnegative constant, i.e., $a(x) \equiv a \in [0, \infty)$.

Lemma 4.3. *Assume $p > P_c^0$, $\beta > 0$, $h_0 \in (0, L_0)$ and $a(x) \equiv a \in [0, \infty)$. Let (u, g, h) be the solution of the problem (1.5) with the maximal existence time $T_{\max}$. If $T_{\max} = \infty$, then (g_∞, h_∞) is a finite interval with length no more than $2L_{c^*}$ and u is bounded. Furthermore, there holds that*

$$\lim_{t\to\infty} \max_{g(t)<x<h(t)} u(t, x) = 0.$$

Proof. Since $T_{\max} = \infty$, one can obtain directly from Corollary 3.3 that (g_∞, h_∞) is a finite interval with length no more than $2L_{c^*}$.

By setting $\alpha = \frac{\beta}{p-1}$ and taking $U(t, x) := e^{\alpha t}u(t, x)$, we transform (1.5) into the following problem

$$\begin{cases} U_t = dU_{xx} + [a + \alpha]U + U^p, & t > 0,\ g(t) < x < h(t), \\ U(t, g(t)) = 0,\ g'(t) = -\mu[e^{-\alpha t}U_x(t, g(t))], & t > 0, \\ U(t, h(t)) = 0,\ h'(t) = -\mu[e^{-\alpha t}U_x(t, h(t))], & t > 0, \\ -g(0) = h_0 = h(0),\ U(0, x) = u_0(x), & -h_0 \le x \le h_0. \end{cases} \tag{4.3}$$

Since $\alpha > 0$, the function $e^{-\alpha t}$ is thus uniformly bounded for all $t > 0$. Furthermore, in view of $-g_\infty$, $h_\infty < \infty$, by introducing the energy functional

$$E[U](t) = \int_{g(t)}^{h(t)} \left[\frac{U_x^2}{2} - \frac{(a+\alpha)}{2}U^2 - \frac{1}{p+1}U^{p+1}\right]\mathrm{d}x,$$

together with rescaling techniques, the almost identical method employed by the proof of [14, Theorem 3.1], [28, Proposition 1 and 2] and [29, Proposition 4.3 and 4.4] yields that U is bounded and there holds that

$$\lim_{t\to\infty} \max_{g(t)<x<h(t)} U(t, x) = 0,$$

which implies that the same result holds for $u(t, x)$. The proof is thus completed. □

Remark 4.4. It is still open whether all of the global solutions of (4.3), furthermore, all of the global solutions of (1.5), are bounded and decay uniformly to zero for the general case that the spatial source $a(x)$ is spatially nontrivial. It seems that one needs more information about the spatial source $a(x)$, in particular, the more concrete spatial dependency pattern of $a(x)$. We leave this issue as an important open problem for the future research.

5. Sharp threshold trichotomy

In what follows, we prove a sharp threshold trichotomy theorem, which, in particular, claims that both the global vanishing solution and global transition solution of the problem (1.5) exist with $T_{\max} = \infty$, depending on whether (g_∞, h_∞) is a finite interval with the length either less than or equivalent exactly to $2L_{c^*}$.

Theorem 5.1. *Suppose $p > 1$, $\beta > 0$, $h_0 \in (0, L_0]$, $\delta > 0$, $\phi \in \mathscr{X}(h_0)$ and $a(x) \equiv a \in [0, \infty)$. Let (u, g, h) be the solution of the problem (1.5) with $u_0(x) = \delta\phi(x)$ and $T_{\max}$ be the maximal existence time. Then there exists $\delta^* = \delta^*(p, \phi) \in [0, \infty)$ such that:*

(i) *(u, g, h) **blows up** in finite time if $\delta > \delta^*$ in the sense that $T_{\max} < \infty$ and*

$$\lim_{t\to T_{\max}} \|u(t, \cdot)\|_{L^\infty([g(t),h(t)])} = \infty; \tag{5.1}$$

(ii) (u,g,h) *is the* ***global vanishing solution*** *if* $\delta<\delta^*$ *in the sense that* $T_{\max}=\infty$, (g_∞,h_∞) *is a finite interval with length less than* $2L_{c^*}$ *and*

$$\lim_{t\to\infty}\max_{g(t)<x<h(t)}u(t,x)=0; \tag{5.2}$$

(iii) (u,g,h) *is the* ***global transition solution*** *if* $\delta=\delta^*$ *in the sense that* $T_{\max}=\infty$, (g_∞,h_∞) *is a finite interval with exact length* $2L_{c^*}$ *and* (5.2) *holds.*

Proof. In view of Corollary 3.3, any solution blows up if $L_{c^*}\le h_0\le L_0$, and hence in such case, it is enough to set $\delta^*=0$ for any $\phi\in\mathscr{X}(h_0)$. In the following proof, we just investigate the remaining case where $0<h_0<L_{c^*}$. Further, to emphasize the dependence of solution on the initial data when necessary, we denote the solution by $(u^\delta,g^\delta,h^\delta)$. So do g^δ_∞, h^δ_∞ and $T^\delta_{\max}$.

To begin with, let

$$\Sigma=\left\{\delta>0:\begin{array}{l}T^\delta_{\max}=\infty\ \text{ and }\ (g^\delta_\infty,h^\delta_\infty)\ \text{ is a finite interval}\\ \text{with length no more than }\ 2L_{c^*}\end{array}\right\}.$$

In view of Theorem 3.1 and Remark 3.2, it follows that Σ is not empty and $\Sigma\subset[0,\Lambda]$, where Λ is defined by (3.5). Set $\delta^*=\delta^*(a,p,\phi)=\sup\Sigma$. Firstly, we claim that $T^{\delta^*}_{\max}=\infty$. In fact, by the continuous dependence, for any fixed $0\le t<T^{\delta^*}_{\max}$, u^δ approaches to u^{δ^*} in $L^\infty((-\infty,\infty))$ as $\delta\uparrow\delta^*$, where we have extended u to 0 on $(-\infty,g(t))\cup(h(t),\infty)$. Since $T^\delta_{\max}=\infty$ for all $0<\delta<\delta^*$, and by Lemma 4.3, any global solution is bounded, it follows that u^δ is bounded for all $0<\delta<\delta^*$, which implies that there exists a constant K such that $\|u^{\delta^*}(t,\cdot)\|_{L^\infty(\mathbb{R})}\le K$ for all $0\le t<T^{\delta^*}_{\max}$. Hence, Theorem 3.4 yields that $T^{\delta^*}_{\max}=\infty$.

Furthermore, we prove $(g^{\delta^*}_\infty,h^{\delta^*}_\infty)$ is a finite interval and $h^{\delta^*}_\infty-g^{\delta^*}_\infty=2L_{c^*}$. In fact, the finiteness of the interval is a direct consequence of the fact that $T^{\delta^*}_{\max}=\infty$. On the other hand, by utilizing the indirect argument, if we assume that $h^{\delta^*}_\infty-g^{\delta^*}_\infty<2L_{c^*}$, then the continuous dependence implies that there exists a sufficiently large $T>0$ such that if ϵ is small enough, the solution $(u^{\delta^*+\epsilon},g^{\delta^*+\epsilon},h^{\delta^*+\epsilon})$ of (1.5) with $u_0=(\delta^*+\epsilon)\phi$ satisfies

$$h^{\delta^*+\epsilon}(T)-g^{\delta^*+\epsilon}(T)<2L_{c^*}.$$

Thus, it follows that $h^{\delta^*+\epsilon}_\infty-g^{\delta^*+\epsilon}_\infty\le 2L_{c^*}$, which contradicts the definition of δ^*.

Thirdly, we show that there only exists a unique δ^* such that $T^{\delta^*}_{\max}=\infty$ and $(g^{\delta^*}_\infty,h^{\delta^*}_\infty)$ is a finite interval with exact length $2L_c$. Otherwise, without loss of generality, we assume that there are two $\delta^*_1>\delta^*_2$ such that the solution $(u^{\delta^*_i},g^{\delta^*_i},h^{\delta^*_i})$ of (1.5) with $u_0=\delta^*_i\phi$ satisfying $T^{\delta^*_i}_{\max}=\infty$ and $(g^{\delta^*_i}_\infty,h^{\delta^*_i}_\infty)$ is a finite interval with exact length $2L_c$ for $i=1$, 2, respectively. The comparison principle implies that for any fixed $T_0>0$,

$$[g^{\delta^*_2}(T_0),h^{\delta^*_2}(T_0)]\subset(g^{\delta^*_1}(T_0),h^{\delta^*_1}(T_0)),$$

and for all $g^{\delta^*_2}(T_0)\le x\le h^{\delta^*_2}(T_0)$, there holds that

$$u^{\delta^*_2}(T_0,x)<u^{\delta^*_1}(T_0,x).$$

Set

$$\Gamma=\left\{\epsilon>0:u^{\delta^*_1}(T_0,x)>u^{\delta^*_2}(T_0,x-\epsilon),\forall x\in[g^{\delta^*_2}(T_0+\epsilon),h^{\delta^*_2}(T_0+\epsilon)]\subset(g^{\delta^*_1}(T_0),h^{\delta^*_1}(T_0))\right\}.$$

Clearly, Γ is bounded. Set $\epsilon_0:=\sup\Gamma$, and define

$$\widetilde{g}(t)=g^{\delta^*_2}(t+T_0)+\epsilon_0,\quad \widetilde{h}(t)=h^{\delta^*_2}(t+T_0)+\epsilon_0$$

and

$$\tilde{u}(t,x)=u^{\delta^*_2}(t+T_0,x-\epsilon_0).$$

It follows that $(\widetilde{u},\widetilde{g},\widetilde{h})$ is the unique solution of (1.5) with

$$\widetilde{u}_0 = u^{\delta_2^*}(T_0, x-\epsilon_0), \quad \widetilde{g}(0) = g^{\delta_2^*}(T_0)+\epsilon_0 \quad \text{and} \quad \widetilde{h}(0) = h^{\delta_2^*}(T_0)+\epsilon_0.$$

In view of the definition of ϵ_0 and the comparison principle, we obtain

$$h_\infty^{\delta_1^*} - g_\infty^{\delta_1^*} \geq \widetilde{h}(\infty) - g_\infty^{\delta_2^*} > 2L_{c^*},$$

which contradicts the definition of δ_1^*. This contradiction derives the desired conclusion.

Finally, combining the proof above together with Corollary 3.3 and Lemma 4.3, we can easily obtain all the rest of conclusions in (**i**)–(**iii**). The proof is finished. □

6. Discussion

In this paper, we investigate a nonlinear free boundary problem incorporating with nontrivial spatial and exponential temporal weighted source. To investigate the asymptotic behavior of the solution, we first establish some sufficient conditions to finite time blowup solution. Furthermore, the existence of global vanishing solutions with sufficiently small initial data is also obtained, which, together with the blowup result, reveals that there exists a Fujita type critical exponent for (1.5) whenever the spatial source is just equivalent to a trivial constant, or is an extreme one, such as “very negative” one in the sense of measure or integral. Finally, we prove a sharp threshold trichotomy result in terms of the size of the initial data, by which the blowup solution, the global vanishing solution, and, in particular, the global transition solution are distinguished.

Moreover, we must note that the conclusions of Lemma 4.3 only hold for the case that the spatial source $a(x)$ is spatially trivial. Thus, the sharp threshold trichotomy theorem, i.e., Theorem 5.1 only hold for the same case. As we have pointed in Remark 4.4, it is still open whether all of the global solutions of (1.5) are bounded and decay uniformly to zero for the general case that the spatial source $a(x)$ is spatially nontrivial. Some hints reveal that more information about the spatial source $a(x)$, in particular, the more concrete spatial dependency pattern of $a(x)$, will be conducive, and we leave this issue as an important open problem for the future research.

On the other hand, it is well known that the Laplace operator is just a symmetric and local operator, however, more and more researches have revealed that by introducing the nonsymmetric operators or nonlocal operators, such as the Laplace operator with the advection term in [30–32] or the nonlocal operator in [33], into the traditional reaction–diffusion, more rich, complicate and interesting dynamics will emerge. Thus, we will also consider such nonsymmetric and nonlocal versions of the problem (1.5) in our future research.

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