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Propagation dynamics for a time-periodic reaction–diffusion SI epidemic model with periodic recruitment

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Abstract. The paper is devoted to the study of the asymptotic speed of spread and traveling wave solutions for a time-periodic reaction–diffusion SI epidemic model which lacks the comparison principle. By using the basic reproduction number R_0 of the corresponding periodic ordinary differential system and the minimal wave speed c^*, the spreading properties of the corresponding solution of the model are established. More precisely, if $R_0 \leqslant 1$, then the solution of the system converges to the disease-free equilibrium as $t \to \infty$ and if $R_0 > 1$, the disease is persistent behind the front and extinct ahead the front. On the basis of it, we then analyze the full information about the existence and nonexistence of traveling wave solutions of the system involved with R_0 and c^*.

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1. Introduction

Considering the influence of seasonal factor on the geographic spread of infectious diseases, in this paper we focus on the following time-periodic reaction–diffusion SI epidemic model with periodic recruitment

$$\begin{cases} \partial_t S(t,x) = d_S \Delta S(t,x) + \mu(t) - \beta(t)\frac{S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \mu(t)S(t,x), \ t>0, \ x \in \mathbb{R}^N, \\ \partial_t I(t,x) = \Delta I(t,x) + \beta(t)\frac{S(t,x)I(t,x)}{S(t,x)+I(t,x)} - r(t)I(t,x), \ t>0, \ x \in \mathbb{R}^N, \\ S(0,x) = S_0(x) \geqslant 0, \ I(0,x) = I_0(x) \geqslant 0, \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $S(t,x)$ and $I(t,x)$ represent the densities of the susceptible individuals and the infective individuals at time t and location x, respectively, d_S represents the diffusion rates of the susceptible individuals and $\beta(t)$ denotes the transmission rate of the infectious disease. For convenience, the recruitment and mortality rates are defined by $\mu(t)$, and $r(t)$ is the removal rate. In addition, the following assumptions can be made:

(A) $\mu(t)$, $\beta(t)$ and $r(t)$ are Hölder continuous and positive nontrivial functions on $\mathbb{R}^+$ and periodic in time with the same period $T > 0$. Furthermore, $S_0(x)$ and $I_0(x)$ are bounded and uniformly continuous functions on $\mathbb{R}^N$.

In the paper, we study the asymptotic speed of spread and traveling wave solutions of model (1.1). Firstly, we investigate the spatial spread of the infection in terms of the basic reproduction number R_0 of the corresponding periodic kinetic system of system (1.1), denoted by $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt}$. Based on it, the complete information about the existence and nonexistence of traveling wave solutions of system (1.1) is shown.

It was reported that the transmission dynamics of infectious diseases can be significantly influenced by the seasonal change, see [2,3,9,13,15,16,25]. Therefore, it is important to investigate the influence of the seasonal change on the geographic spread of infectious diseases. Here, we mention that the major difficulty to study (1.1) is that system (1.1) is not monotone. Thus, the theory on the spreading speeds and traveling wave solutions for monotone semiflows (see [10,19,20,33]) is not applicable to (1.1). Recently, the asymptotic speed of spread and traveling waves solutions for autonomous epidemic models has been studied by using various methods, see [5,7,12,17,22,24,26,28–31,34,38,40] and the references therein. However, the study for the asymptotic speed of spread and traveling waves solutions of nonautonomous epidemic models is few. Up to now, the main progress is due to [1,27,32,37]. In [32], Wang et al. firstly investigated the following time-periodic reaction–diffusion SI model

$$\begin{cases} \partial_t S(t,x) = d_1 \Delta S(t,x) - \beta(t)\frac{S(t,x)I(t,x)}{S(t,x)+I(t,x)},\ t>0,\ x\in\mathbb{R},\\ \partial_t I(t,x) = d_2 \Delta I(t,x) + \beta(t)\frac{S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(t)I(t,x),\ t>0,\ x\in\mathbb{R}, \end{cases} \tag{1.2}$$

and established the existence and nonexistence of periodic traveling waves of system (1.2). In [37], they further studied a time-periodic reaction–diffusion SI model with bilinear incidence by using the method developed by [36].

Different from (1.2), the vital dynamics is incorporated in (1.1); namely, there are a periodic recruitment term and the natural mortality terms in (1.1). For the autonomous SI disease model with vital dynamics, the asymptotic speed of spread and traveling wave solutions has been investigated by Ducrot and Magal [8], Ducrot [4], Ducrot et al. [6], Li and Zou [18], Zhao and Wang [39], Zhao et al [41] and the references therein. More recently, Ambrosio et al. [1] considered generalized traveling waves for a nonautonomous SI disease model with vital dynamics and bilinear incidence in a general time-heterogeneous environment. Except for [1], there seem no results on the asymptotic speed of spread and traveling wave solutions for nonautonomous SIR disease model with vital dynamics. Therefore, the purpose of this paper is to investigate the asymptotic speed of spread and traveling wave solutions for system (1.1).

The rest of this paper is organized as follows. In the next section, we investigate spreading properties of the corresponding solution for system (1.1). In Sect. 3, we are concerned with the existence and nonexistence of traveling wave solutions of (1.1) for $(t,x)\in\mathbb{R}^2$.

2. Spreading properties

In this section, we take into account the spreading properties of solutions of system (1.1). Before stating the main results, we need some suitable estimates of solutions of system (1.1). Let $\mathbb{X} = BUC(\mathbb{R}^N,\mathbb{R}^2)$ be the Banach space of bounded and uniformly continuous functions from $\mathbb{R}^N$ to $\mathbb{R}^2$, which is endowed with the usual supremum norm. Its positive cone $\mathbb{X}^+$ consists of all functions in $\mathbb{X}$ with both nonnegative components.

Lemma 2.1. *Let* (A) *be satisfied. Then system (1.1) generates a strongly continuous and globally defined semiflow on* $\mathbb{X}^+$ *defined by* $\{T(t)\}_{t\geqslant 0}$ *or for each* $U_0{=}(S_0,I_0){\in}\mathbb{X}^+$ *as* $\{T(t)U_0 = (S(t,\cdot;U_0), I(t,\cdot;U_0))\}_{t\geqslant 0}$. *For each initial data* $U_0=(S_0,I_0)\in\mathbb{X}^+$, *the solution* $(S,I)(t,x;U_0)=(S,I)(t,x)$ *of system (1.1) satisfies the following properties:*

(i) $(S,I)\in C^{1+\frac{\theta}{2},2+\theta}((0,\infty)\times\mathbb{R})\bigcap C([0,\infty);\mathbb{X}^+)$ *for some* $\theta\in(0,1)$;
(ii) For each $t\geqslant 0$ *and* $x\in\mathbb{R}^N$, *it holds that:*

$$S(t,x)\leqslant 1+e^{-\int_0^t \mu(s)ds}\left(\|S_0\|_\infty - 1\right),\ t\geqslant 0,\ x\in\mathbb{R}^N; \tag{2.1}$$

(iii) There exists a positive constant B_v *such that for any* $U_0=(S_0,I_0)\in\mathbb{X}^+$, *one has*

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{R}} I(t,x)\leqslant B_v.$$

Proof. It is easy to see that (i) and (ii) hold true. Thus, we prove only (iii). Due to (ii), there is a positive integer l large enough such that $0 \leqslant S(t,x) \leqslant \frac{3}{2}$ for each $t \geqslant lT$ and $x \in \mathbb{R}^N$. Let $\eta := \sup_{x\in\mathbb{R}^N} I(lT,x) < \infty$ by (A). Since $I(t,x)$ satisfies

$$\partial_t I(t,x) \leqslant \Delta I(t,x) + \left(\frac{\frac{3}{2}\beta(t)}{\frac{3}{2}+I(t,x)} - r(t)\right) I(t,x),\ \forall t > lT,\ x \in \mathbb{R}^N,$$

it follows from the comparison principle that one has

$$I(t,x) \leqslant V(t;\eta),\ \forall t > lT,\ x \in \mathbb{R}^N, \tag{2.2}$$

where the $V(t;\eta)$ is the solution of the following ordinary differential equations

$$\begin{cases} \frac{dV}{dt} = \left(\frac{\frac{3}{2}\beta(t)}{\frac{3}{2}+V(t)} - r(t)\right) V(t),\ t > lT, \\ V(lT) = \eta. \end{cases} \tag{2.3}$$

It is obvious that there exists a positive constant B_v independent upon η such that there is an integer $l_v > l$ satisfying $V(t;\eta) \leqslant B_v,\ \forall t \geqslant l_v T$, which means that conclusion (iii) holds. It completes the proof. $\square$

Now we firstly analyze the dynamics of (1.1) when $R_0 \leqslant 1$. The following theorem means that if $R_0 \leqslant 1$, then the solution of the system converges to the disease-free equilibrium as $t \to \infty$, that is, the disease uniformly dies out.

Theorem 2.2. *Let (A) be satisfied. If* $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} \leqslant 1$, *then for each initial data* $(S_0, I_0) \in \mathbb{X}^+$, *the solution of system (1.1) satisfies*

$$\lim_{t\to\infty} (S,I)(t,x) = (1,0) \quad \text{uniformly with respect to } x \in \mathbb{R}^N.$$

Proof. It is obvious that $R_0 \leqslant 1$ implies that $\frac{1}{T}\int_0^T \big(\beta(t) - r(t)\big)dt \leqslant 0$. Let $q(t,V) = \frac{\frac{3}{2}\beta(t)}{\frac{3}{2}+V} - r(t)$, then one has $\int_0^T q(t,0)dt = \int_0^T \big(\beta(t) - r(t)\big)dt \leqslant 0$. Thus, [42, Theorem 3.1.2] associated with (2.3) implies that $\lim_{t\to\infty} V(t;\eta) = 0$. It follows from (2.2) that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} I(t,x) = 0. \tag{2.4}$$

For every $\epsilon > 0$ with $\epsilon \in \left(0, \min_{t\in\mathbb{R}} \frac{\mu(t)}{2\beta(t)}\right)$, it follows from (2.4) that there exists a positive integer $n_\epsilon \in N$ sufficiently large such that $I(t,x) < \epsilon$ for $t > n_\epsilon T$ and $x \in \mathbb{R}^N$. Consider the following equation with parameter $\epsilon > 0$:

$$\begin{cases} \partial_t S^\epsilon(t,x) = d_S \Delta S^\epsilon(t,x) + \mu(t) - \mu(t)S^\epsilon(t,x) - \beta(t)\epsilon,\ \forall t > 0,\ x \in \mathbb{R}^N, \\ S^\epsilon(0,x) = S(n_\epsilon T, x),\ x \in \mathbb{R}^N. \end{cases}$$

It follows from [35, Lemma 2.1] that equation

$$\frac{du^\epsilon(t)}{dt} = \mu(t) - \mu(t)u^\epsilon(t) - \beta(t)\epsilon$$

admits a unique positive T-periodic solution $u^\epsilon(t)$ which is globally attractive in $[0,\infty)$. Then by the comparison principle we have that

$$\lim_{t\to\infty} S^\epsilon(t,x) = u^\epsilon(t) \ \text{ uniformly in } x \in \mathbb{R}^N.$$

Since $S(t,x)$ satisfies

$$\partial_t S(t,x) \geqslant d_S \Delta S(t,x) + \mu(t) - \mu(t)S(t,x) - \beta(t)\epsilon,\ \forall t \geqslant n_\epsilon T,\ x \in \mathbb{R}^N,$$

it follows from the comparison principle that

$$S(t,x) \geqslant S^\epsilon(t - n_\epsilon T, x),\ t > n_\epsilon T,\ x \in \mathbb{R}^N.$$

Therefore, one has

$$\liminf_{t\to\infty} S(t,x) \geqslant \liminf_{t\to\infty} u^{\epsilon}(t) \ \text{ uniformly in } x\in\mathbb{R}^N.$$

A direct computation gives $\lim_{\epsilon\to 0}\inf_{t\in\mathbb{R}} u^{\epsilon}(t)$=1. Then by the arbitrariness of ϵ, we have $\liminf_{t\to\infty} S(t,x) \geqslant 1$ uniformly in $x\in\mathbb{R}^N$. Moreover, (2.1) implies that $\limsup_{t\to\infty} S(t\ ,x)\leqslant 1$ uniformly in $x\in\mathbb{R}^N$. As a consequence, one has $\lim_{t\to\infty} S(t,x)=1$ uniformly in $x\in\mathbb{R}^N$. This completes the proof. □

Next, we take into account the spread properties of solution of (1.1) when $R_0>1$ and $c\in(-c^*,c^*)$, where $c^*=2\sqrt{\frac{1}{T}\int_0^T\Big(\beta(t)-r(t)\Big)dt}$. To solve it, we firstly analyze a weak uniform persistence property in Lemma 2.4 and then establish the uniform persistence property in Theorem 2.5. The following lemma will be used in the proof of Lemma 2.4.

Lemma 2.3. *Let $a(t)$ be the Hölder continuous function and periodic in time t with a period $T>0$. For each $\mathcal{L}>0$, $c\in\mathbb{R}$ and $e\in\mathbb{S}^{N-1}$, let $\lambda_{\mathcal{L}}[c,e]$ be the principle eigenvalue of parabolic eigenvalue problem*

$$\begin{cases}\partial_t u(t,x)-\Delta u(t,x)+ce\cdot\nabla u(t,x)+a(t)u(t,x)=\lambda_{\mathcal{L}}[c,e]u(t,x),\ \ x\in B(0,\mathcal{L}),\\ u(t,x)=0,\ \ x\in\partial B(0,\mathcal{L}),\\ u(t,x)>0,\ \ x\in B(0,\mathcal{L}),\\ u(t,x)=u(t+T,x),\ \ x\in B(0,\mathcal{L}).\end{cases}$$

Then $\lambda_{\mathcal{L}}[c,e]$ does not depend upon $e\in\mathbb{S}^{N-1}$, it is denoted by $\lambda_{\mathcal{L}}[c]$ and one has

$$\lim_{\mathcal{L}\to\infty}\lambda_{\mathcal{L}}[c]=\frac{c^2}{4}+\bar{a}$$

locally uniformly for $c\in\mathbb{R}$, where $\bar{a}=\frac{1}{T}\int_0^T a(t)dt$.

Proof. Let $\psi(t,x)=u(t,x)e^{-\alpha e\cdot x}$ with $\alpha\in\mathbb{R}$. By a straightforward computation, one has $\nabla u(t,x)=(\nabla\psi+\alpha e\psi)e^{\alpha e\cdot x}$ and $\Delta u(t,x)=(\Delta\psi(t,x)+2\alpha e\cdot\nabla\psi+\alpha^2\psi)e^{\alpha e\cdot x}$. Then $\psi(t,x)$ satisfies

$$\begin{aligned}&\partial_t\psi(t,x)+ce\cdot\nabla\psi(t,x)+\alpha c\psi(t,x)-\Delta\psi(t,x)\\&\qquad -2\alpha e\cdot\nabla\psi(t,x)-\alpha^2\psi(t,x)+a(t)\psi(t,x)=\lambda_{\mathcal{L}}[c,e]\psi(t,x).\end{aligned}$$

Choose now $c=2\alpha$. Then one has

$$\begin{cases}\partial_t\psi(t,x)-\Delta\psi(t,x)+(\frac{c^2}{4}+a(t))\psi(t,x)=\lambda_{\mathcal{L}}[c,e]\psi(t,x),\\ \psi(t,x)=0,\ \ \forall x\in\partial B(0,\mathcal{L}),\\ \psi(t,x)>0,\forall x\in B(0,\mathcal{L}).\end{cases}$$

According to [14], one gets $\lambda_{\mathcal{L}}[c,e]=\gamma_{\mathcal{L}}+\frac{c^2}{4}+\bar{a}$, where $\bar{a}=\frac{1}{T}\int_0^T a(t)dt$ and $\gamma_{\mathcal{L}}$ is the principle eigenvalue of the following elliptic problem

$$\begin{cases}-\Delta\psi(x)=\gamma\psi(x),\ \ \forall x\in B(0,\mathcal{L}),\\ \psi(x)=0,\ \ \forall x\in\partial B(0,\mathcal{L}),\end{cases}$$

with the corresponding principle eigenfunction $\psi(x)$. In particular, one has

$$\gamma_{\mathcal{L}}=\inf_{\varphi\in H_0^1(B(0,\mathcal{L}))}\frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}=\frac{1}{\mathcal{L}^2}\gamma_1.$$

Therefore, $\lambda_{\mathcal{L}}[c,e]=\frac{1}{\mathcal{L}^2}\gamma_1+\frac{c^2}{4}+\bar{a}$. This completes the proof. □

Lemma 2.4. *Assume that* $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$. *Let* $z > 0$ *and* $c_0 \in [0, c^*)$ *be given. Then there exists* $\epsilon = \epsilon(z, c_0) > 0$ *such that for each* $x \in \mathbb{R}^N$, $c \in [-c_0, c_0]$ *and* $U_0 \in M^z \times (M^z \backslash \{0\})$, *there holds that*

$$\limsup_{t\to\infty} I(t, x + cte, U_0) \geqslant \epsilon,$$

where $M^z = \{\phi \in BUC(\mathbb{R}^N, \mathbb{R}) : \ 0 \leqslant \phi \leqslant z\}$.

Proof. We prove the lemma by a contradiction. On the contrary, we suppose that for each $n \in \mathbb{N}$, there exist $U_0^n = (S_0^n, I_0^n) \in M^z \times (M^z \setminus \{0\})$, $x_n \in \mathbb{R}^N$, $c_n \in [-c_0, c_0]$ and $e_n \in \mathbb{S}^{N-1}$ such that the solution of system (1.1) with initial value U_0^n, denoted by (S^n, I^n), satisfies

$$\limsup_{t\to\infty} I^n(t, x_n + c_n te_n) \leqslant \frac{1}{n+1}.$$

As a consequence, for any $n \in \mathbb{N}$, there exists a positive integer N_n sufficiently large such that

$$I^n(t + N_nT, x_n + c_n(t + N_nT)e_n) \leqslant \frac{2}{n+1}, \ \forall t \geqslant 0.$$

Consider the sequence of functions u_n and v_n defined by

$$\begin{aligned} u_n(t,x) =& S^n(t + N_nT, x_n + x + c_n(t + N_nT)e_n),\\ v_n(t,x) =& I^n(t + N_nT, x_n + x + c_n(t + N_nT)e_n), \ \forall n \in \mathbb{N}, \ t \geqslant 0, \ x \in \mathbb{R}^N. \end{aligned} \tag{2.5}$$

It is obvious that

$$v_n(t, 0) \leqslant \frac{2}{n+1}, \ \forall t \geqslant 0, \ n \in \mathbb{N}. \tag{2.6}$$

Here, we make the following claim: One has

$$\lim_{n\to\infty} (u_n, v_n)(t, x) = (1, 0) \tag{2.7}$$

uniformly on $t \geqslant 0$ and x in bounded sets.

Before proving the claim, we firstly complete the proof of the theorem. Due to $0 \leqslant c_0 < c^*$, we take $\eta > 0$ small enough such that

$$\frac{(c_0)^2}{4} + \frac{\eta}{T}\int_0^T \beta(t)dt < \frac{(c^*)^2}{4}. \tag{2.8}$$

Due to Lemma 2.3 and (2.8), we can obtain that there exists $L = L_\eta > 0$ such that the principle eigenvalue $\lambda_L[c_n]$ of the following problem

$$\begin{cases} \partial_t u - \Delta u + c_n e_n \cdot \nabla u(t,x) + a_\eta(t)u(t,x) = \lambda_L[c_n]u(t,x), \ \forall t \in \mathbb{R}, \ x \in B(0,L),\\ u(t,x) = 0, \ \forall t \in \mathbb{R}, \ x \in \partial B(0,L),\\ u(t,x) > 0, \ \forall t \in \mathbb{R}, \ x \in B(0,L),\\ u(t,x) = u(t+T,x), \ \forall t \in \mathbb{R}, \ x \in B(0,L), \end{cases} \tag{2.9}$$

satisfies $\lambda_L[c_n] < 0, \ \forall n \geqslant 1$, where $a_\eta(t) := -[\beta(t)(1-\eta) - r(t)]$ and

$$\lambda_L[c_n] = \frac{\gamma_1}{L^2} + \frac{c_n^2}{4} + \bar{a}_\eta.$$

Following the claim, we conclude that

$$\lim_{n\to\infty} u_n(t,x) = 1 \text{ and } \lim_{n\to\infty} v_n(t,x) = 0$$

uniformly on $t > 0$ and $x \in B(0, L)$. Therefore, there exists $n_\eta > 0$ such that

$$1 - \eta \leqslant u_n(t,x) \leqslant 1 + \eta, \ 0 < v_n(t,x) \leqslant \eta, \ \forall t \geqslant 0, \ x \in B(0,L), \ n \geqslant n_\eta.$$

It follows from (2.5) that

$$(\partial_t + c_n e_n \cdot \nabla - \Delta)\, v_n(t,x) \geqslant \Big(\beta(t)(1-\eta) - r(t)\Big) v_n(t,x)$$

for all $n \geqslant n_\eta$, $t \geqslant 0$ and $x \in B(0,L)$. That is,

$$(\partial_t + c_n e_n \cdot \nabla - \Delta)\, v_n(t,x) + a_\eta v_n(t,x) \geqslant 0,\ \forall n \geqslant n_\eta,\ t \geqslant 0,\ x \in B(0,L).$$

Let $n \geqslant n_\eta$ be fixed. Let Θ be the principle eigenfunction of (2.9). Consider also $\delta > 0$ small enough such that $v_n(0,x) \geqslant \delta\Theta(0,x),\ \forall x \in B(0,L)$.

Define an operator $Q := \partial_t + c_n e_n \cdot \nabla - \Delta + a_\eta(t)$. Then we have got

$$Q[v_n](t,x) \geqslant 0,\ \forall (t,x) \in [0,\infty) \times B(0,L).$$

Since the function $\underline{v}(t,x) = \delta e^{-\lambda_L[c_n]t}\Theta(t,x)$ satisfies

$$\begin{aligned} &Q[\underline{v}(t,x)] = 0,\ \forall (t,x) \in [0,\infty) \times B(0,L),\\ &\underline{v}(0,x) = \delta\Theta(0,x) \leqslant v_n(0,x),\ \forall x \in B(0,L),\\ &\underline{v}(t,x) = 0 \leqslant v_n(t,x),\ \forall t \geqslant 0,\ \forall x \in \partial B(0,L), \end{aligned}$$

the parabolic comparison principle implies $\underline{v}(t,x) \leqslant v_n(t,x)$ for $t \geqslant 0,\ x \in B(0,L)$. Due to $\lambda_L[c_n] < 0$, we obtain $v_n(t,0) \to \infty$ as $t \to \infty$, which leads to a contradiction with (2.6).

Now, we are in the position to prove the claim, namely, we show that (2.7) holds. Due to Lemma 2.1 and parabolic estimates, one may assume that $(u_n, v_n) \to (u_\infty, v_\infty)$ locally uniformly for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ (up to subsequence). Since $\mathbb{S}^{N-1}$ is compact, we assume that $e_n \to e \in \mathbb{S}^{N-1}$. According to $\{c_n\}_{n\geqslant 0} \subset [-c_0, c_0]$, one may also assume that $c_n \to c \in [-c_0, c_0]$ as $n \to \infty$. Due to the periodicity of $\beta(t)$, $\mu(t)$ and $r(t)$ and Lemma 2.1, (u_∞, v_∞) satisfies

$$\begin{cases} 0 \leqslant u_\infty(t,x) \leqslant 1,\ 0 \leqslant v_\infty(t,x) \leqslant B_v,\\ (\partial_t + ce\cdot\nabla - d_S\Delta)\, u_\infty(t,x) = \mu(t) - \beta(t)\frac{u_\infty(t,x)v_\infty(t,x)}{u_\infty(t,x)+v_\infty(t,x)} - \mu(t)u_\infty(t,x),\\ (\partial_t + ce\cdot\nabla - \Delta)\, v_\infty(t,x) = \beta(t)\frac{u_\infty(t,x)v_\infty(t,x)}{u_\infty(t,x)+v_\infty(t,x)} - r(t)v_\infty(t,x),\\ u_\infty(t,x) = u_\infty(t+T,x),\ v_\infty(t,x) = v_\infty(t+T,x) \end{cases}$$

for any $(t,x) \in \mathbb{R}^{N+1}$. Furthermore, (2.6) leads to $v_\infty(t,0) = 0$ for any $t \geqslant 0$, which combining the parabolic maximum principle yields

$$v_\infty(t,x) \equiv 0 \text{ and } u_\infty(t,x) \equiv 1.$$

Give $L > 0$. Assume by contradiction that $v_n \to 0$ as $n \to \infty$ but not uniformly on $[0,\infty)\times\overline{B(0,L)}$, which means that there exist a sequence $(t_n, x_n) \in [0,\infty)\times\overline{B(0,L)}$ and $\epsilon > 0$ such that $v_n(t_n,x_n) \geqslant \epsilon$. Without loss of generality, let $x_n \to x_\infty \in \overline{B(0,L)}$ as $n \to \infty$. Let $t_n' = t_n - [t_n/T]T \in [0,T)$. For a convenience, one also assume $t_n' \to t_0 \in [0,T]$ as $n \to \infty$. Consider the sequence of functions $w_n(t,x) = v_n(t + [t_n/T]T, x)$. By Lemma 2.1 and the parabolic estimates, one assumes that it converges locally uniformly to some function $w_\infty(t,x)$ as $n \to \infty$. In particular, $w_\infty(t_0, x_\infty) \geqslant \epsilon$.

Using (2.5), (2.6) and the periodicity of $\beta(t)$, $\mu(t)$ and $r(t)$, one obtains that w_∞ satisfies

$$\begin{cases} w_\infty(t_0, 0) = 0,\\ (\partial_t - \Delta + ce\cdot\nabla)\, w_\infty(t,x) = a(t,x)w_\infty(t,x), \end{cases}$$

where $a \equiv a(t,x)$ is some given bounded function. It follows from the strong maximum principle that $w_\infty(t,x_\infty) \equiv 0$, which contradicts $w_\infty(t_0,x_\infty) \geqslant \epsilon$. Following the above arguments, we have

$$\lim_{n\to\infty} \sup_{t\geqslant 0, x\in B(0,L)} v_n(t,x) = 0,\ \forall L > 0.$$

Next, we show that $u_n \to 1$ uniformly for $t \geqslant 0$ and locally uniformly in $x \in \mathbb{R}^N$ by the contradiction way. Set $L > 0$ be given and assume that there exist $\epsilon > 0$ and a sequence $(t_n, x_n) \in [0, \infty) \times \overline{B(0, L)}$ such that

$$|1 - u_n(t_n, x_n)| \geqslant \epsilon.$$

Let $t'_n = t_n - [t_n/T]T \in [0, T)$. Without loss of generality, we assume $x_n \to x_\infty \in \overline{B(0, L)}$ and $t'_n \to t_0 \in [0, T]$. Let $w^*_n(t, x) = u_n(t + [t_n/T]T, x)$. Then one has that w^*_n converges to some function w^*_∞ as $n \to \infty$. It is clear that

$$|1 - w^*_\infty(t_0, x_\infty)| \geqslant \epsilon. \tag{2.10}$$

In particular, w^*_∞ is a bounded entire solution of

$$\begin{cases} w^*_\infty(t + T, x) = w^*_\infty(t, x), \\ (\partial_t + ce \cdot \nabla - d_S \Delta)\, w^*_\infty = \mu(t) - \mu(t) w^*_\infty. \end{cases} \tag{2.11}$$

It follows from Lemma 2.1 that $0 \leqslant w^*_\infty(t, x) \leqslant 1$ on $\mathbb{R} \times \mathbb{R}^N$. Then (2.10) implies that

$$0 \leqslant \inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} w^*_\infty(t, x) \leqslant 1 - \epsilon.$$

Since $w^*_\infty(t, x)$ is T-periodic in $t \in \mathbb{R}$, there exists a sequence $\{(\hat{t}_n, \hat{x}_n)\}_{n \in \mathbb{N}} \subset [0, T) \times \mathbb{R}^N$ such that

$$\lim_{n \to \infty} w^*_\infty(\hat{t}_n, \hat{x}_n) = \inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} w^*_\infty(t, x).$$

Possibly up to a subsequence, we assume that $\hat{t}_n \to \hat{t}_\infty \in [0, T]$ as $n \to +\infty$. For each $n \in \mathbb{N}$, define a function $w^*_{\infty,n}(t, x) := w^*_\infty(t, x + \hat{x}_n)$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Then due to the parabolic estimate, there exists some function $\tilde{w}^*_\infty(t, x)$ such that $w^*_{\infty,n}$ converges (up to a subsequence) to $\tilde{w}^*_\infty$ locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ as $n \to \infty$ and $\tilde{w}^*_\infty$ satisfies

$$\begin{cases} (\partial_t + ce \cdot \nabla - d_S \Delta)\, \tilde{w}^*_\infty = \mu(t) - \mu(t) \tilde{w}^*_\infty, \\ \tilde{w}^*_\infty(\hat{t}_\infty, 0) = \min\limits_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \tilde{w}^*_\infty(t, x) \leqslant 1 - \epsilon. \end{cases} \tag{2.12}$$

Plugging $(\hat{t}_\infty, 0)$ into the first equation of (2.12) yields $(\partial_t + ce \cdot \nabla - d_S \Delta) \tilde{w}^*_\infty(\hat{t}_\infty, 0) \leqslant 0$ and $\mu(\hat{t}_\infty) \big(1 - \tilde{w}^*_\infty (\hat{t}_\infty, 0)\big) > 0$. There is a contradiction. Therefore, one has $u_n \to 1$ uniformly for $t \geqslant 0$ and locally uniformly in $x \in \mathbb{R}^N$. This completes the proof of the claim. □

Now we are in the position to establish the uniform persistence of the solution of (1.1) when $R_0 > 1$ and $c \in (-c^*, c^*)$.

Theorem 2.5. *Let* $R_0 := \frac{\int_0^T \beta(t) dt}{\int_0^T r(t) dt} > 1$, $c \in [0, c^*)$ *and* $z > 0$ *be given. Then there exists* $\hat{\epsilon} = \hat{\epsilon}(c, z) > 0$ *such that for each* $U_0 \in M^z \times (M^z \setminus \{0\})$, $x \in \mathbb{R}^N$ *and* $e \in \mathbb{S}^{N-1}$, *we have*

$$\liminf_{t \to \infty} I(t, x + cte) \geqslant \hat{\epsilon}.$$

Proof. Let us argue by a contradiction. Assume that there exists a sequence of initial data $\{U^m_0 = (S^m_0, I^m_0)\}_{m \geqslant 0} \subset M^z \times (M^z \setminus \{0\})$, $\{x_m\}_{m \geqslant 0} \subset \mathbb{R}^N$ and $\{e_m\}_{m \geqslant 0} \subset \mathbb{S}^{N-1}$ such that the sequence of solutions of (1.1) denoted by (S^m, I^m) satisfies

$$\liminf_{t \to \infty} I^m(t, x_m + cte_m; U^m_0) \leqslant \frac{1}{m+1}, \ \forall m \geqslant 0.$$

Let $\epsilon = \epsilon(c, z) > 0$ be the constant provided by Lemma 2.4. It is clear that

$$\limsup_{t \to \infty} I(t, x + cet; U_0) \geqslant \epsilon$$

for each $U_0 \in M^z \times (M^z \setminus \{0\})$, $x \in \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$.

Next we define $W^m(t,x) = S^m(t, x_m + x + ce_m t; U_0^m)$ and $V^m(t,x) = I^m(t, x_m + x + ce_m t; U_0^m)$. Then there exists a sequence $\{t_m\}_{m\in\mathbb{N}}$ satisfying $t_m \to \infty$ as $m \to \infty$ and a sequence $\{a_m\}_{m\geqslant 0} \in (0,\infty)$ such that

$$V^m(t_m, 0) = \frac{\epsilon}{2},\ V^m(t,0) \leqslant \frac{\epsilon}{2},\ \forall t \in [t_m, t_m + a_m],$$
$$V^m(t_m + a_m, 0) \leqslant \frac{1}{m+1}.$$

Let $t'_m = t_m - [t_m/T]T \in [0,T)$ and $N_m = [t_m/T]$. For convenience, we also assume that $t'_m \to t'_0 \in [0,T]$ as $m \to \infty$. Up to a subsequence, one may assume that $V^m(t + N_mT, x) \to V^\infty(t,x)$ and $W^m(t + N_mT, x) \to W^\infty(t,x)$ locally uniformly for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ as $m \to \infty$. Furthermore, the function V^∞ satisfies

$$V^\infty(t'_0, 0) = \frac{\epsilon}{2},\ V^\infty(t,0) \leqslant \frac{\epsilon}{2},\ \forall t \in [t'_0, t'_0 + \tilde{L}),$$

where $\tilde{L} = \liminf_{m\to\infty} a_m$. On the other hand, one may assume that $e_m \to e \in \mathbb{S}^{N-1}$ as $m \to \infty$ so that functions W^∞ and V^∞ satisfy

$$\begin{cases} (\partial_t + ce\cdot\nabla - d_S\Delta)\, W^\infty = \mu(t) - \beta(t)\frac{W^\infty V^\infty}{W^\infty + V^\infty} - \mu(t)W^\infty, & (t,x) \in \mathbb{R}^{1+N}, \\ (\partial_t + ce\cdot\nabla - \Delta)\, V^\infty = \beta(t)\frac{W^\infty V^\infty}{W^\infty + V^\infty} - r(t)V^\infty, & (t,x) \in \mathbb{R}^{1+N}. \end{cases}$$

If $\tilde{L} < \infty$, then one obtains $V^\infty(t'_0 + \tilde{L}, 0) = 0$. Consequently, the strong comparison principle implies that $V^\infty(t,x) \equiv 0$ for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, which contradicts the fact $V^\infty(t'_0, 0) = \frac{\epsilon}{2}$.

If $\tilde{L} = \infty$, which means that $a_m \to \infty$ as $m \to \infty$, one has

$$V^\infty(t,0) \leqslant \frac{\epsilon}{2},\ \forall t \in [t'_0, \infty). \tag{2.13}$$

Now recall that functions (S^∞, I^∞) defined by

$$S^\infty(t,x) = W^\infty(t, x - cet),\ I^\infty(t,x) = V^\infty(t, x - cet)$$

satisfy the system

$$\begin{cases} (\partial_t - d_S\Delta)\, S^\infty = \mu(t) - \beta(t)\frac{S^\infty I^\infty}{S^\infty + I^\infty} - \mu(t)S^\infty, \\ (\partial_t - \Delta)\, I^\infty = \beta(t)\frac{S^\infty I^\infty}{S^\infty + I^\infty} - r(t)I^\infty. \end{cases}$$

Since $(S^\infty, I^\infty) \in M^z \times (M^z \setminus \{0\})$, it follows that $\limsup\limits_{t\to\infty} I^\infty(t, cet) \geqslant \epsilon$. Recalling to $I^\infty(t, cet) \equiv V^\infty(t, 0)$, one has

$$\limsup_{n\to\infty} V^\infty(t,0) \geqslant \epsilon,$$

which contradicts (2.13). It completes the proof. □

Following the above theorem, we have the following corollary which is a convergence result.

Corollary 2.6. *Let* $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$, $z > 0$, $c \in (-c^*, c^*)$, $e \in \mathbb{S}^{N-1}$ *and* $U_0 = (S_0, I_0) \in (M^z \times M^z \setminus \{0\})$ *be given. Let* $\{t_n\}_{n\in\mathbb{N}}$ *satisfy* $t_n \to \infty$ *as* $n \to \infty$. *Then there exists a subsequence, still denoted by* $\{t_n\}_{n\in\mathbb{N}}$, *such that*

$$\lim_{n\to\infty} (S,I)(t + [t_n/T]T, x + c(t + [t_n/T]T)e; U_0) = (S^\infty, I^\infty)(t, x - cet)$$

locally uniformly for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ *and where* (S^∞, I^∞) *is a bounded entire solution of (1.1) such that* $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} I^\infty(t,x) > 0$.

Proof. Define $W_n(t,x) = S(t+[t_n/T]T, x+c(t+[t_n/T]T)e)$ and $V_n(t,x) = I(t+[t_n/T]T, x+c(t+[t_n/T]T)e)$. Using Lemma 2.1 and the standard parabolic estimates, up to a subsequence, we assume that $\{(W_n, V_n)\}$ converges to some (W,V) locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, which satisfies the following system

$$\begin{cases}(\partial_t + ce\cdot\nabla - d_S\Delta)\,W = \mu(t) - \beta(t)\frac{WV}{W+V} - \mu(t)W, & (t,x)\in\mathbb{R}\times\mathbb{R}^N,\\ (\partial_t + ce\cdot\nabla - \Delta)\,V = \beta(t)\frac{WV}{W+V} - r(t)V, & (t,x)\in\mathbb{R}\times\mathbb{R}^N.\end{cases}$$

In view of Theorem 2.5, we have that there exists $\epsilon>0$ such that $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} V(t,x)\geqslant\epsilon$. Note that $(S^\infty, I^\infty)(t,x)\equiv(U,V)(t,x+cet)$ is an entire solution of (1.1). This completes the proof. □

In the following, we consider the case that $R_0>1$ and $\|x\|\geqslant ct$ with $c\geqslant c^*$.

Theorem 2.7. *Assume that* $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$ *and* $z>0$ *be given. Let* $U_0=(S_0,I_0)\in M^z\times(M^z\setminus\{0\})$ *be given such that* I_0 *is compactly supported. Then for each* $\frac{\alpha c^*}{2}<\frac{N}{2}$ *with* $\alpha>0$*, there is*

$$\limsup_{t\to\infty, |x|\geqslant c^*t-\alpha\ln t}\,[I(t,x)+|S(t,x)-1|]=0.$$

Proof. It is easy to see that $\hat{I}(t,x)=e^{\int_0^t(\beta(s)-r(s))ds}(T_\Delta(t)I_0)(x)$ satisfies the following linear system

$$\begin{cases}(\partial_t-\Delta)\,\hat{I}(t,x)=(\beta(t)-r(t))\,\hat{I}(t,x),\ \forall t\geqslant 0,\ x\in\mathbb{R}^N,\\ \hat{I}(0,x)=\hat{I}_0(x),\ x\in\mathbb{R}^N.\end{cases}$$

Due to (1.1) and (A), one has

$$(\partial_t-\Delta)\,I(t,x)\leqslant(\beta(t)-r(t))\,I(t,x),\ x\in\mathbb{R}^N.$$

It follows from the strong maximal principle that $I(t,x)\leqslant\hat{I}(t,x)$ for all $t\geqslant 0$ and $x\in\mathbb{R}^N$. On the other hand, we have

$$\int_{\mathbb{R}^N} e^{-\frac{\|x+cet-y\|^2}{4t}} I_0(y)dy\leqslant e^{-\frac{c^2t}{4}}e^{-\frac{c(e,x)}{2}}J(c),$$

where $J(c)=\int_{\mathbb{R}^N} e^{\frac{c}{2}|y|}I_0(y)dy$. Since I_0 is compactly supported, $J(c)<\infty$ is well defined. Using similar arguments to those in [4, Lemma 5.6], we obtain that if $\frac{\alpha c^*}{2}<\frac{N}{2}$, then for each $x\in\mathbb{R}^N$ and each $t>0$ such that $\|x\|\geqslant c^*t-\alpha\ln t$, we have

$$I(t,x)\leqslant(4\pi t)^{-\frac{N}{2}}t^{\frac{\alpha c^*}{2}}J(c^*),$$

which implies that

$$\limsup_{t\to\infty, |x|\geqslant c^*t-\alpha\ln t} I(t,x)=0. \tag{2.14}$$

Next we show that

$$\limsup_{t\to\infty, |x|\geqslant c^*t-\alpha\ln t}|S(t,x)-1|=0$$

by a contradiction. Assume on the contrary that there exist $\epsilon>0$ and a sequence $\{(t_n,x_n)\}_{n\in\mathbb{N}}$, where $t_n\to\infty(n\to\infty)$ and $\{x_n\}_{n\in\mathbb{N}}\in\mathbb{R}^N$, satisfying

$$\begin{cases}\|x_n\|\geqslant c^*t_n-\alpha\ln t_n,\ \forall n\in\mathbb{N},\\ |S(t_n,x_n)-1|\geqslant\epsilon,\ \forall n\in\mathbb{N}.\end{cases}$$

Let $t'_n = t_n - [t_n/T]T \to t'_0 \in [0,T]$ as $n \to \infty$ and $N_n = [t_n/T]$. Let $S_n(t,x) = S(t + N_nT, x + x_n)$ and $I_n(t,x) = I(t+N_nT, x+x_n)$. According to the parabolic estimates, one can assume, up to a subsequence, that they converge to some entire solution (S_∞, I_∞) of (1.1) which satisfies

$$|S_\infty(t'_0, 0) - 1| \geqslant \epsilon. \tag{2.15}$$

Using (2.14), we get $I_\infty(t'_0, 0) = 0$, which combining the strong maximal principle further implies that $I_\infty(t,x) \equiv 0$ on $\mathbb{R} \times \mathbb{R}^N$. As a consequence, the function S_∞ is an entire solution of the following equation

$$(\partial_t - d_S\Delta)S_\infty(t,x) = \mu(t)\left(1 - S_\infty(t,x)\right), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

We have $S_\infty(t,x) \equiv 1$ on $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ by using the strong maximal principle, which contradicts the fact (2.15). This completes the proof. □

Remark 2.8. Theorems 2.5 and 2.7 mean that the disease is persistent when $R_0 > 1$ and $\|x\| = ct$ with $c \in [0, c^*)$, and the disease is extinct when $R_0 > 1$ and $\|x\| = ct$ with $c \geqslant c^*$. In summary, in this section we showed that if $R_0 \leqslant 1$, then the disease uniformly dies out (Theorem 2.2), and if $R_0 > 1$, the disease is persistent behind the front and extinct ahead the front (Theorems 2.5 and 2.7).

3. Periodic traveling waves

In this section, we focus on the time-periodic traveling waves of (1.1) in $\mathbb{R} \times \mathbb{R}$ (namely, $N = 1$). When $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$, we first establish the existence of nontrivial periodic traveling waves of (1.1) with speed $c > c^* > 0$ by using the method developed by [32,36,37] recently and then show the asymptotical behaviors of traveling wave solutions at infinity by using the spreading properties established in Sect. 3. Furthermore, we show the existence of a minimal periodic traveling wave solution with speed c^* by a limiting argument. Finally, we also show the nonexistence of traveling wave solutions when $R_0 \leq 1$ or $R_0 > 1$ and $c < c^*$. Here, we recall that

$$c^* = 2\sqrt{\frac{1}{T}\int_0^T (\beta(t) - r(t))dt}.$$

A time T-periodic traveling wave solution of (1.1) is defined to be a solution of the form

$$\begin{aligned} &S(t,x) = u(t, x+ct) := u(t,z), \ I(t,x) = v(t, x+ct) := v(t,z), \ \forall (t,z) \in \mathbb{R} \times \mathbb{R} \\ &\text{and } u(t,z) = u(t+T, z), \ v(t,z) = v(t+T, z), \ \forall (t,z) \in \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{3.1}$$

By substituting (3.1) into (1.1), we can get the following wave form equations

$$\begin{cases} u_t(t,z) = d_S u_{zz}(t,z) - cu_z(t,z) + \mu(t) - \beta(t)\frac{u(t,z)v(t,z)}{u(t,z)+v(t,z)} - \mu(t)u(t,z), \\ v_t(t,z) = v_{zz}(t,z) - cv_z(t,z) + \beta(t)\frac{u(t,z)v(t,z)}{u(t,z)+v(t,z)} - r(t)v(t,z). \end{cases} \tag{3.2}$$

We are looking for time-periodic traveling wave solutions for system (1.1), which are positive solutions of system (3.2) and supplemented with the following boundary conditions

$$u(t,-\infty) = 1, \ v(t,-\infty) = 0; \ \liminf_{z\to\infty} u(t,z) > 0, \ \liminf_{z\to\infty} v(t,z) > 0 \tag{3.3}$$

uniformly for $t \in \mathbb{R}$.

Linearizing the second equation in system (3.2) at the disease-free equilibrium $(1,0)$ yields

$$v_t(t,z) = v_{zz}(t,z) - cv_z(t,z) + \beta(t)v(t,z) - r(t)v(t,z). \tag{3.4}$$

Letting $v(t,z) := \mathcal{Q}(t)e^{\lambda z}$ and plugging it into (3.4), we get the characteristic equation as follows:

$$\frac{d\mathcal{Q}(t)}{dt} = \left(\lambda^2 - c\lambda + \beta(t) - r(t)\right)\mathcal{Q}(t).$$

From a straightforward computation, it further follows that

$$\mathcal{Q}(t) = \exp\left(\int_0^t [\lambda^2 - c\lambda + \beta(s) - r(s)]ds\right).$$

Due to the T-periodicity of $\beta(t)$ and $r(t)$, we have that the T-periodicity of the function $\mathcal{Q}(t)$ is equivalent to

$$\lambda^2 - c\lambda + \kappa = 0, \tag{3.5}$$

where $\kappa := \frac{1}{T}\int_0^T (\beta(t) - r(t))dt$. Thus, the two roots of (3.5) are given by $\lambda_c := \frac{c-\sqrt{c^2-4\kappa}}{2}$ and $\lambda_2 = \frac{c+\sqrt{c^2-4\kappa}}{2}$ if $c > 2\sqrt{\kappa}$ and $R_0 > 1$. Here, we emphasize that $c^* = 2\sqrt{\kappa}$.

In the following, we assume that $R_0 > 1$, $c > c^* := 2\sqrt{\kappa}$ and

$$\mathcal{V}(t) = \exp\left(\int_0^t [\lambda_c^2 - c\lambda_c + \beta(s) - r(s)]ds\right).$$

Based on the above arguments, we can obtain the following lemmas.

Lemma 3.1. *The function $v^+(t,z) := \mathcal{V}(t)e^{\lambda_c z}$ satisfies the following equation*

$$v_t^+ = v_{zz}^+ - cv_z^+ + (\beta(t) - r(t))\,v^+.$$

Lemma 3.2. *Assume that ϵ_1 is sufficiently small with $0 < \epsilon_1 < \lambda_c$ and $\mathcal{M}$ is large enough. Then the function $u^-(t,z) := \max\{1 - \mathcal{M}e^{\epsilon_1 z}, 0\}$ satisfies*

$$u_t^- - d_S u_{zz}^- + cu_z^- - \mu(t) + \mu(t)u^- \leqslant -\frac{\beta(t)v^+u^-}{v^+ + u^-}, \quad \forall z \neq z_1 := -\epsilon_1^{-1}\ln\mathcal{M}. \tag{3.6}$$

Proof. If $z > -\epsilon_1^{-1}\ln\mathcal{M}$, then $u^-(t,z) = 0$, and hence, (3.6) is valid.

If $z < -\epsilon_1^{-1}\ln\mathcal{M}$, then $u^-(t,z) = 1 - \mathcal{M}e^{\epsilon_1 z}$. Thus, we need only to prove that

$$\mathcal{M}\epsilon_1 e^{\epsilon_1 z}(c - d_S\epsilon_1) + \mu(t)\mathcal{M}e^{\epsilon_1 z} \geqslant \frac{\beta(t)\mathcal{V}(t)e^{\lambda_c z}(1 - \mathcal{M}e^{\epsilon_1 z})}{\mathcal{V}(t)e^{\lambda_c z} + (1 - \mathcal{M}e^{\epsilon_1 z})},\ z < -\epsilon_1^{-1}\ln\mathcal{M}.$$

It is sufficient to verify

$$\mathcal{M}\epsilon_1(c - d_S\epsilon_1) \geqslant \beta(t)\mathcal{V}(t)e^{-\epsilon_1^{-1}(\lambda_c-\epsilon_1)\ln\mathcal{M}} = \beta(t)\mathcal{V}(t)\mathcal{M}^{-\epsilon_1^{-1}(\lambda_c-\epsilon_1)}.$$

The above inequality holds provided $\mathcal{M} = \frac{1}{\epsilon_1}$ with ϵ_1 small enough. $\square$

Lemma 3.3. *Suppose that θ with $0 < \theta < \min\{\epsilon_1, \lambda_2 - \lambda_c\}$ is sufficiently small and $\mathcal{J} > 0$ with $-\theta^{-1}\ln\mathcal{J} < -\epsilon_1^{-1}\ln\mathcal{M}$ is large enough. Then the function $v^-(t,z) := \max\{\mathcal{V}(t)e^{\lambda_c z}(1 - \mathcal{J}e^{\theta z}), 0\}$ satisfies*

$$v_t^- - v_{zz}^- + cv_z^- \leqslant -r(t)v^- + \mathcal{Q}[u^-, v^-]$$

for any $z \neq z_2 := -\theta^{-1}\ln\mathcal{J}$, where $\mathcal{Q}[u^-, v^-]$ is defined by

$$\mathcal{Q}[u,v] = \begin{cases} 0, & \text{if } u(t,z)v(t,z) = 0,\ \forall(t,z) \in \mathbb{R}\times\mathbb{R}, \\ \frac{\beta(t)u(t,z)v(t,z)}{u(t,z)+v(t,z)}, & \text{if } u(t,z)v(t,z) \neq 0,\ \forall(t,z) \in \mathbb{R}\times\mathbb{R}. \end{cases}$$

Proof. By similar arguments to the proof of [32, Lemma 2.3], we can prove the lemma. Thus, we omit it. $\square$

Let $N > -z_2$ and $C_N := C(\mathbb{R} \times [-N, N], \mathbb{R}^2)$. Define a convex cone $\mathcal{D}_N$ as

$$\mathcal{D}_N = \left\{ (\bar{u}, \bar{v}) \in C_N \left| \begin{array}{l} \bar{u}(t,z) = \bar{u}(t+T,z),\ \forall (t,z) \in \mathbb{R} \times [-N,N], \\ \bar{v}(t,z) = \bar{v}(t+T,z),\ \forall (t,z) \in \mathbb{R} \times [-N,N], \\ u^-(t,z) \leqslant \bar{u}(t,z) \leqslant 1,\ \forall (t,z) \in \mathbb{R} \times [-N,N], \\ v^-(t,z) \leqslant \bar{v}(t,z) \leqslant \min\{v^+(t,z), \Lambda\},\ \forall (t,z) \in \mathbb{R} \times [-N,N], \\ \bar{u}(t, \pm N) = u^-(t, \pm N),\ \forall t \in \mathbb{R}, \\ \bar{v}(t, \pm N) = v^-(t, \pm N),\ \forall t \in \mathbb{R} \end{array} \right. \right\},$$

where $\Lambda := \max_{t \in \mathbb{R}} \frac{\beta(t)}{r(t)}$. For any given $(\bar{u}, \bar{v}) \in \mathcal{D}_N$, consider the initial value problem as follows:

$$\begin{cases} u_t - \mathcal{A}_1 u = f_1[\bar{u}, \bar{v}],\ t > 0,\ z \in [-N, N], \\ v_t - \mathcal{A}_2 v = f_2[\bar{u}, \bar{v}],\ t > 0,\ z \in [-N, N], \\ u(0,z) = u_0(z),\ v(0,z) = v_0(z),\ z \in [-N,N],\ u_0, v_0 \in C([-N,N]), \\ u(t, \pm N) = \bar{G}_u(t, \pm N),\ v(t, \pm N) = \bar{G}_v(t, \pm N),\ \forall t > 0, \end{cases} \tag{3.7}$$

where

$$\begin{aligned} &\mathcal{A}_1 u = d_S \partial_{zz} u - c \partial_z u - \alpha_1 u,\ \mathcal{A}_2 v = \partial_{zz} v - c \partial_z v - \alpha_2 v, \\ &f_1[\bar{u}, \bar{v}] := \alpha_1 \bar{u} + \mu(t) - \mathcal{Q}[\bar{u}, \bar{v}](t,z) - \mu(t)\bar{u},\ f_2[\bar{u}, \bar{v}] := \alpha_2 \bar{v} + \mathcal{Q}[\bar{u}, \bar{v}](t,z) - r(t)\bar{v}, \\ &\alpha_1 > \max_{t \in [0,T]} \{\beta(t) + \mu(t)\},\ \alpha_2 > \max_{t \in [0,T]} r(t) \end{aligned}$$

and

$$\bar{G}_u(t,z) := \frac{1}{2} u^-(t,-N) - \frac{z}{2N} u^-(t,-N),\ \bar{G}_v(t,z) := \frac{1}{2} v^-(t,-N) - \frac{z}{2N} v^-(t,-N)$$

for any $t \in [0,T]$ and $z \in [-N,N]$. It is easy to see that $\bar{G}_u(t, \pm N) = u^-(t, \pm N)$ and $\bar{G}_v(t, \pm N) = v^-(t, \pm N)$ for $t \in \mathbb{R}$ and the functions $\bar{G}_u$ and $\bar{G}_v$ are T-periodic and belong to $C^{1,2}(\mathbb{R} \times [-N,N])$. Set $\tilde{u}(t,z) = u(t,z) - \bar{G}_u(t,z)$, $\tilde{v}(t,z) = v(t,z) - \bar{G}_v(t,z)$, $\tilde{F}_u = \mathcal{A}_1 \bar{G}_u(t,z) - \partial_t \bar{G}_u(t,z)$ and $\tilde{F}_v = \mathcal{A}_2 \bar{G}_v(t,z) - \partial_t \bar{G}_v(t,z)$. Then problem (3.7) reduces to

$$\begin{cases} \partial_t \tilde{u}(t,z) - \mathcal{A}_1 \tilde{u}(t,z) = f_1[\bar{u}, \bar{v}] + \tilde{F}_u,\ t > 0,\ z \in [-N, N], \\ \partial_t \tilde{v}(t,z) - \mathcal{A}_2 \tilde{v}(t,z) = f_2[\bar{u}, \bar{v}] + \tilde{F}_v,\ t > 0,\ z \in [-N, N], \\ \tilde{u}(0,z) = u_0(z) - \bar{G}_u(0,z),\ \tilde{v}(0,z) = v_0(z) - \bar{G}_u(0,z),\ z \in [-N,N], \\ \tilde{u}(t, \pm N) = 0,\ \tilde{v}(t, \pm N) = 0,\ t > 0. \end{cases} \tag{3.8}$$

The realization of $\mathcal{A}_i$ in $C([-N,N])$ with the homogenous Dirichlet boundary condition can be defined by

$$D(A_i^0) = \left\{ w \in \textstyle\bigcap_{p \geqslant 1} W^{2,p}_{loc}((-N,N)) : w, \mathcal{A}_i w \in C([-N,N]), w|_{\pm N} = 0 \right\},$$
$$A_i^0 w = \mathcal{A}_i w,\ i = 1, 2.$$

In fact, $D(\mathcal{A}_i) = \left\{ u \in C^2([-N,N]), u|_{\pm N} = 0 \right\}$ (see, e.g., [21, Section 5.1.2]). Assume that $\{T_i(t)\}_{t \geqslant 0}$ are the strongly continuous analytic semigroup generated by $A_i^0 : D(A_i^0) \subset C([-N,N]) \to C([-N,N])$ (see [21]). Note that

$$T_i(t) w(x) = e^{-\alpha_i t} \int_{-N}^{N} \Gamma_i(t,x,y) w(y) dy,\ i = 1, 2,\ w(x) \in C([-N,N])$$

for $t > 0$ and $x \in [-N, N]$, where Γ_1 and Γ_2 are the Green function associated with $d_S\partial_{xx} - c\partial_x$ and $\partial_{xx} - c\partial_x$ and Dirichlet boundary condition, respectively. Then system (3.8) can be treated as the following integral system

$$\begin{cases} \tilde{u}(t,z) = T_1(t)\tilde{u}(0)(z) + \int_0^t T_1(t-s)\big(f_1[\bar{u},\bar{v}](s) + \tilde{F}_u(s)\big)(z)ds, \\ \tilde{v}(t,z) = T_2(t)\tilde{v}(0)(z) + \int_0^t T_2(t-s)\big(f_2[\bar{u},\bar{v}](s) + \tilde{F}_v(s)\big)(z)ds, \end{cases}$$

where $t \geqslant 0$ and $z \in [-N, N]$. Then $(u(t,z), v(t,z))$ satisfies

$$\begin{cases} u(t,z) = T_1(t)\tilde{u}(0)(z) + \int_0^t T_1(t-s)\big(f_1[\bar{u},\bar{v}](s) + \tilde{F}_u(s)\big)(z)ds + \bar{G}_u(t,z), \\ v(t,z) = T_2(t)\tilde{v}(0)(z) + \int_0^t T_2(t-s)\big(f_2[\bar{u},\bar{v}](s) + \tilde{F}_v(s)\big)(z)ds + \bar{G}_v(t,z), \end{cases} \tag{3.9}$$

where $t \geqslant 0$ and $z \in [-N, N]$. A solution of (3.9) can be called as a mild solution of (3.8). Note that $f_i[\bar{u},\bar{v}] \in C(\mathbb{R} \times [-N,N])$ and $f_i[\bar{u},\bar{v}](t,\cdot) \in C([-N,N])$; then it follows from [21, Theorem 5.1.17] that the functions u and v defined by (3.9) belong to $C([0,2T] \times [-N,N]) \bigcap C^{\theta,2\theta}([\epsilon,2T] \times [-N,N])$ for every $\epsilon \in (0,2T)$ and $\theta \in (0,1)$. Define a set

$$\mathcal{D}_N^0 = \left\{ (u_0,v_0) \in C([-N,N],\mathbb{R}^2) \,\middle|\, \begin{array}{l} u^-(0,z) \leqslant u_0(z) \leqslant u^+(0,z),\ \forall z \in [-N,N], \\ v^-(0,z) \leqslant v_0(z) \leqslant \min\{v^+(0,z),\Lambda\},\ \forall z \in [-N,N], \\ u_0(\pm N) = u^-(0,\pm N), \\ v_0(\pm N) = v^-(0,\pm N), \end{array} \right\}.$$

It is easy to see that $\mathcal{D}_N^0$ is a closed and convex set.

Lemma 3.4. *For any $(u_0,v_0) \in \mathcal{D}_N^0$, let $(u_N(t,z;u_0,v_0), v_N(t,z;u_0,v_0))$ be the solutions of system (3.9) with the initial value (u_0,v_0). Then*

$$u^-(t,x) \leqslant u_N(t,z;u_0,v_0) \leqslant 1,\ \ v^-(t,x) \leqslant v_N(t,z;u_0,v_0) \leqslant \max\{v^+(t,x),\Lambda\}$$

for any $(t,x) \in [0,\infty) \times [-N,N]$.

Proof. The argumentations are essentially same as those in [36, Lemma 3.3] and [37, Lemma 2.4], so we omit them. □

For a given $(\bar{u},\bar{v}) \in \mathcal{D}_N$, define a map $F_{(\bar{u},\bar{v})} : \mathcal{D}_N^0 \to C([-N,N],\mathbb{R}^2)$ by

$$F_{(\bar{u},\bar{v})}[u_0,v_0](\cdot) = (u_N(T,\cdot;u_0,v_0), v_N(T,\cdot;u_0,v_0)),$$

where $(u_N(T,\cdot;u_0,v_0), v_N(T,\cdot;u_0,v_0))$ is the solution of system (3.7). In view of Lemma 3.4 and the periodicity of u^-, u^+, v^- and v^+, we have $F_{(\bar{u},\bar{v})}[\mathcal{D}_N^0] \in \mathcal{D}_N^0$. Obviously, $\mathcal{D}_N^0$ is a complete metric space with a distance induced by the supremum norm. For any (u_0^1,v_0^1) and $(u_0^2,v_0^2) \in \mathcal{D}_N^0$, it follows from (3.9) that

$$\begin{aligned} \|u_N(T,\cdot;u_0^1,v_0^1) - u_N(T,\cdot;u_0^2,v_0^2)\| = & \sup_{z\in[-N,N]} |e^{-\alpha_1 T} \int_{-N}^{N} \Gamma_1(T,x,y)\big(u_0^1 - u_0^2\big)dy| \\ \leqslant & e^{-\alpha_1 T}\|u_0^1 - u_0^2\|_{C([-N,N])}. \end{aligned}$$

On the same way,

$$\|v_N(T,\cdot;u_0^1,v_0^1) - v_N(T,\cdot;u_0^2,v_0^2)\| \leqslant e^{-\alpha_2 T}\|v_0^1 - v_0^2\|_{C([-N,N])}.$$

Since $e^{-\alpha_i T} < 1$ for $i = 1, 2$, one has that $F_{(u,v)} : \mathcal{D}_N^0 \to \mathcal{D}_N^0$ is a contraction map. As a consequence, the Banach fixed point theorem implies that $F_{(\bar{u},\bar{v})}$ admits a unique fixed point $(u_0^*, v_0^*) \in \mathcal{D}_N^0$. Let $(\bar{u}_N^*(t,z), \bar{v}_N^*(t,z)) = (u_N(t,z;u_0^*,v_0^*), v_N(t,z;u_0^*,v_0^*))$ for any $t \in [0,\infty)$ and $z \in [-N,N]$, where $(u_N(t,z;u_0^*,v_0^*), v_N(t,z;u_0^*,v_0^*))$ is the solution of system (3.7) with initial value (u_0^*, v_0^*). In view of $(u_0^*(z), v_0^*(z)) = (u_N(T,z;u_0^*,v_0^*), v_N(T,z;u_0^*,v_0^*))$, we get $(\bar{u}_N^*(t+T,z), \bar{v}_N^*(t+T,z)) = (\bar{u}_N^*(t,z), \bar{v}_N^*(t,z))$ for all $t \in [0,\infty)$ and $z \in [-N,N]$. Define $(\bar{u}_N^*(t,z), \bar{v}_N^*(t,z)) = (\bar{u}_N^*(t-kT,z), \bar{v}_N^*(t-kT,z))$ for $t \in \mathbb{R}$ and $z \in [-N,N]$, where $k \in \mathbb{Z}$ satisfies $kT \leqslant t \leqslant (k+1)T$. Then $(\bar{u}_N^*(t,z), \bar{v}_N^*(t,z)) = (\bar{u}_N^*(t+T,z), \bar{v}_N^*(t+T,z))$ for all $t \in [0,\infty)$ and $z \in [-N,N]$. According to Lemma 3.4, it is easy to see that $(\bar{u}_N^*(t,z), \bar{v}_N^*(t,z)) \in \mathcal{D}_N$. Moreover, $(\bar{u}_N^*(t,z), \bar{v}_N^*(t,z))$ satisfies

$$\begin{cases} \bar{u}_N^*(t) = T_1(t-s)(\bar{u}_N^*(s) - \bar{G}_u(s)) + \int\limits_s^t T_1(t-m)\big(f_1[\bar{u}_N^*, \bar{v}_N^*](m) + \tilde{F}_u(m)\big)dm + \bar{G}_u(t), \\ \bar{v}_N^*(t) = T_2(t-s)(\bar{v}_N^*(s) - \bar{G}_v(s)) + \int\limits_s^t T_2(t-m)\big(f_2[\bar{u}_N^*, \bar{v}_N^*](m) + \tilde{F}_v(m)\big)dm + \bar{G}_v(t), \end{cases} \tag{3.10}$$

for any $t \geqslant s$. On the basis of the above discussion, we obtain the following theorem.

Theorem 3.5. *For any given $(\bar{u}, \bar{v}) \in \mathcal{D}_N$, there exists a unique solution $(\bar{u}_N^*, \bar{v}_N^*) \in \mathcal{D}_N$ so that (3.10) holds.*

By virtue of Theorem 3.5, we can define an operator $\mathcal{E} : \mathcal{D}_N \to \mathcal{D}_N$ by $\mathcal{E}(\bar{u}, \bar{v}) = (\bar{u}_N^*, \bar{v}_N^*)$. In what follows, by using the similar arguments to [36, Lemma 3.5] and [37, Lemma 2.6], we present the complete continuity of the operator $\mathcal{E}$ without proof.

Lemma 3.6. *The operator $\mathcal{E} : \mathcal{D}_N \to \mathcal{D}_N$ is completely continuous.*

Based on the above arguments, the Schauder's fixed point theorem implies that $\mathcal{E}$ admits a fixed point $(\hat{u}_N^*, \hat{v}_N^*) \in \mathcal{D}_N$. In addition, $(\hat{u}_N^*(t+T,\cdot), \hat{v}_N^*(t+T,\cdot)) = (\hat{u}_N^*(t,\cdot), \hat{v}_N^*(t,\cdot))$ for all $t \in \mathbb{R}$. Note that $\hat{u}_N^*, \hat{v}_N^* \in C^{\frac{\theta}{2},\theta}(\mathbb{R} \times [-N,N])$ for some $\theta \in (0,1)$. By [21, Theorem 5.1.18 and 5.1.19], it follows that $\hat{u}_N^*, \hat{v}_N^* \in C^{1,2}(\mathbb{R} \times [-N,N])$ satisfy

$$\begin{cases} \partial_t \hat{u}_N^* = d_S \partial_{zz} \hat{u}_N^* - c\partial_z \hat{u}_N^* + \mu(t) - \beta(t)\frac{\hat{u}_N^* \hat{v}_N^*}{\hat{u}_N^* + \hat{v}_N^*} - \mu(t)\hat{u}_N^*, \ t \in \mathbb{R}, \ z \in [-N,N], \\ \partial_t \hat{v}_N^* = \partial_{zz} \hat{v}_N^* - c\partial_z \hat{v}_N^* + \beta(t)\frac{\hat{u}_N^* \hat{v}_N^*}{\hat{u}_N^* + \hat{v}_N^*} - r(t)\hat{v}_N^*, \ t \in \mathbb{R}, \ z \in [-N,N], \\ \hat{u}^*(t, \pm N) = u^-(t, \pm N), \ \hat{v}^*(t, \pm N) = v^-(t, \pm N), \ \forall t \in \mathbb{R}. \end{cases} \tag{3.11}$$

Similar to [36, Theorem 3.6] and [37, Theorem 2.7], we have the following local uniform estimates on $\hat{u}_N^*$ and $\hat{v}_N^*$.

Lemma 3.7. *Let $p \geqslant 2$. For any given $L > 0$, there exists a constant $C(p,L) > 0$ such that for $N > \max\{L, -z_2\}$ large enough, then*

$$\|\hat{u}_N^*\|_{W_p^{1,2}([0,T]\times[-L,L])}, \|\hat{v}_N^*\|_{W_p^{1,2}([0,T]\times[-L,L])} \leqslant C.$$

In addition, there exists a constant $\hat{C}(L) > 0$ such that, for any $z_0 \in \mathbb{R}$,

$$\|\hat{u}_N^*\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T]\times[z_0-L,z_0+L])}, \|\hat{v}_N^*\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T]\times[z_0-L,z_0+L])} \leqslant \hat{C}$$

for $N > \max\{L + |z_0|, -z_2\}$, where $\theta \in (0,1)$.

Now we are in a position to show the main results in this section.

Theorem 3.8. *Assume that $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$. For any $c > c^*$, system (1.1) admits a time-periodic traveling wave solution (u^*, v^*) satisfying (3.2) and*

$$\lim_{z\to-\infty} u^*(t,z) = 1, \qquad \inf_{t\in\mathbb{R}, z\in\mathbb{R}} u^*(t,z) \geq \Lambda_1, \qquad \lim_{z\to-\infty} v^*(t,z) = 0, \qquad \liminf_{z\to\infty} v^*(t,z) \geq \Lambda_2$$

uniformly for $t \in \mathbb{R}$. In particular, both Λ_1 and Λ_2 are independent of $c > c^*$.

Proof. Firstly, we show the existence of the periodic traveling waves for (1.1). Let $\{N_m\}_{m\geqslant 1}$ be an increasing sequence such that $N_m \geqslant -z_2$ for $m \geq 1, m \in \mathbb{N}$ and $\lim_{m\to\infty} N_m = \infty$. It then follows that the solution $(\hat{u}^*_{N_m}, \hat{v}^*_{N_m}) \in \mathcal{D}_{N_m}$ satisfies Lemma 3.7 and (3.11). By virtue of the periodicity of the solution $(\hat{u}^*_{N_m}, \hat{v}^*_{N_m})$ with $t \in \mathbb{R}$, we can extract a subsequence of $(\hat{u}^*_{N_m}, \hat{v}^*_{N_m})$, still denoted by $(\hat{u}^*_{N_m}, \hat{v}^*_{N_m})$, converging toward a function $(u^*, v^*) \in C_{loc}(\mathbb{R}^2)$ in the following topologies

$$(\hat{u}^*_{N_m}, \hat{v}^*_{N_m}) \to (u^*, v^*) \text{ in } C^{\frac{1+\beta}{2},1+\beta}_{loc}(\mathbb{R}^2), \text{ in } H^1_{loc}(\mathbb{R}^2) \text{ and in } L^2_{loc}(\mathbb{R}, H^2_{loc}(\mathbb{R}^2)) \text{ weakly},$$

where $\beta \in (0, \theta)$ and $\theta \in (0, 1)$. Obviously,

$$(u^*, v^*) \in C^{\frac{1+\beta}{2},1+\beta}_{loc}(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2) \cap L^2_{loc}(\mathbb{R}, H^2_{loc}(\mathbb{R}^2)).$$

Since $(\hat{u}^*_{N_m}, \hat{v}^*_{N_m})$ is T-periodic in t, we have

$$(u^*(t + T, z), u^*(t + T, z)) = (u^*(t, z), u^*(t, z)),\ (t, z) \in \mathbb{R}^2$$

and hence, for any $L > 0$ there exists a positive constant $C_0(L)$ such that

$$\|u^*\|_{C^{\frac{1+\beta}{2},1+\beta}_{[0,T]\times[-L,L]}(\mathbb{R}^2)} + \|v^*\|_{C^{\frac{1+\beta}{2},1+\beta}_{[0,T]\times[-L,L]}(\mathbb{R}^2)} \leqslant C_0$$

due to Lemma 3.7. Consequently, similar to [36, Theorem 3.6] and [37, Theorem 2.9], we have that (u^*, v^*) is in $C^{1+\frac{\nu}{2},2+\nu}(\mathbb{R}^2)$ for some $\nu \in (0, 1)$ and satisfies

$$\begin{cases} \partial_t u^* = d_S \partial_{zz} u^* - c\partial_z u^* + \mu(t) - \beta(t)\frac{u^*v^*}{u^*+v^*} - \mu(t)u^*,\ (t, z) \in \mathbb{R}^2, \\ \partial_t v^* = \partial_{zz} v^* - c\partial_z v^* + \beta(t)\frac{u^*v^*}{u^*+v^*} - r(t)v^*,\ (t, z) \in \mathbb{R}^2. \end{cases}$$

In addition, it follows from $(\hat{u}^*_{N_m}, \hat{v}^*_{N_m}) \in \mathcal{D}_{N_m}$ that

$$u^-(t, z) \le u^*(t, z) \le 1, \quad v^-(t, z) \le v^*(t, z) \le \min\{v^+(t, z), \Lambda\}, \quad \forall (t, z) \in \mathbb{R}^2. \tag{3.12}$$

Next, we intend to verify that (u^*, v^*) satisfies the boundary conditions (3.3). Using (3.12) and the definitions of u^- and $v^\pm$, one easily gets

$$u^* \to 1,\ v^* \to 0 \quad \text{uniformly in } t \in \mathbb{R} \quad \text{as } z \to -\infty.$$

Since $u^*(t, z)$ satisfies

$$\begin{aligned} &\partial_t u^* \ge d_S \partial_{zz} u^* - c\partial_z u^* + \mu(t) - (\beta(t) + \mu(t))u^*,\ (t, z) \in \mathbb{R}^2, \\ &u^*(t, z) > 0, \quad u^*(t, z) = u^*(t + T, z) \quad (t, z) \in \mathbb{R}^2, \end{aligned}$$

it is easy to show that

$$\inf_{(t,z)\in\mathbb{R}^2} u^*(t, z) \ge \Lambda_1. \tag{3.13}$$

where $\Lambda_1 := \min_{t\in\mathbb{R}} \frac{\mu(t)}{\beta(t)+\mu(t)}$. Fix $\bar{c} \in [0, c^*)$. It follows from Theorem 2.5 that there exists a constant $\hat{\epsilon} > 0$ such that

$$\liminf_{t\to\infty} v^*(t, x + ct + \bar{c}t) \geqslant \hat{\epsilon},\ \forall x \in \mathbb{R}. \tag{3.14}$$

By (3.14), we have $\liminf_{n\to\infty} v^*(nT, (c+\bar{c})nT) \ge \hat{\epsilon}$, which together with the periodicity of $v^*(t, z)$ on t implies $\liminf_{n\to\infty} v^*(0, (c + \bar{c})nT) \geqslant \hat{\epsilon}$. Now applying the Harnack inequality for the parabolic equations [11,32], we conclude that there exists a constant $\epsilon > 0$ such that $\liminf_{z\to\infty} v^*(t, z) \ge \epsilon$ uniformly in $t \in \mathbb{R}$. In view of (3.13), we further have that $v^*(t, z)$ satisfies

$$\begin{aligned} &\partial_t v^* \ge \partial_{zz} v^* - c\partial_z v^* + \left(\frac{\beta(t)\Lambda_1}{\Lambda_1 + v^*(t, z)} - r(t)\right) v^*(t, z),\ (t, z) \in \mathbb{R}^2, \\ &v^*(t, z) > 0, \quad v^*(t, z) = v^*(t + T, z) \quad (t, z) \in \mathbb{R}^2. \end{aligned}$$

Note that $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$. Then by [42, Theorem 3.1.2], the following periodic ordinary differential equation

$$u'(t) = \left(\frac{\beta(t)\Lambda_1}{\Lambda_1 + u(t)} - r(t) \right) u(t)$$

admits a unique positive T-periodic solution $u^{\triangle}(t)$ and $u^{\triangle}(t)$ is globally asymptotically stable. Consider the following periodic reaction–diffusion equation

$$\partial_t U = \partial_{xx} U + \left(\frac{\beta(t)\Lambda_1}{\Lambda_1 + U(t,x)} - r(t) \right) U(t,x), \quad t > 0,\ x \in \mathbb{R}. \tag{3.15}$$

With the aid of general results in [23], we conclude that for any $c > c^*$, equation (3.15) admits a traveling waves $\phi(t, x + ct) = \phi(t, z)$ satisfying

$$\phi(t, -\infty) = 0, \quad \phi(t, +\infty) = u^{\triangle}(t), \quad \phi(t + T, z) = \phi(t, z), \quad \phi(t, z) > 0$$

for $(t, z) \in \mathbb{R}^2$, and for any $U_0(z)$ satisfying

$$U_0(z) \sim Ae^{\lambda_c z}, \quad \text{as } z \to -\infty, \qquad \liminf_{z \to +\infty} U_0(z) > 0,$$

there holds

$$\lim_{t \to \infty} \frac{|U(t, x; U_0) - \phi(t, x + ct)|}{\phi(t, x + ct)} = 0. \tag{3.16}$$

Based on the above analysis for $v^*(t, z)$, it follows that $\psi(t, x) := v^*(t, x + ct)$ satisfies

$$\partial_t \psi \geq \partial_{xx} \psi + \left(\frac{\beta(t)\Lambda_1}{\Lambda_1 + \psi(t,x)} - r(t) \right) \psi(t,x),\ (t,x) \in \mathbb{R}^2,$$

and hence, the comparison principle implies that

$$v^*(t, x + ct) = \psi(t, x) \geq U(t, x; v^*(0, \cdot)), \quad \forall t \geq 0,\ x \in \mathbb{R}. \tag{3.17}$$

Thus, it follows from (3.16) and (3.17) that

$$v^*(t, z) \geq \phi(t, z) - \frac{1}{2} \min_{t \in \mathbb{R}} u^{\triangle}(t), \qquad \forall\ z \in \mathbb{R} \text{ and } t \geq nT$$

for some $n \in \mathbb{N}$ large enough. Furthermore, we can conclude that

$$\liminf_{z \to +\infty} v^*(t, z) \geq \frac{1}{2} \min_{t \in \mathbb{R}} u^{\triangle}(t) =: \Lambda_2 \quad \text{uniformly in } t \in \mathbb{R}.$$

This completes the proof. □

Theorem 3.9. *Assume that* $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$. *For* $c = c^*$, *system (1.1) admits a time-periodic traveling wave solution* (u, v) *satisfying (3.2) and (3.3).*

Proof. Let $\{c_m\} \in (c^*, c^* + 1)$ be a decreasing sequence such that $\lim_{m \to \infty} c_m = c^*$. It follows Theorem 3.8 that for each c_m, there exists a solution (u_m^*, v_m^*) of (3.2)–(3.3). By a shift, it is always assumed that

$$v_m^*(0, 0) = \frac{1}{2}\Lambda_2, \qquad v_m^*(0, z) < \frac{1}{2}\Lambda_2, \qquad z < 0.$$

Without loss of generality, we can extract a subsequence of $\{(u_m^*, v_m^*)\}_{m \in \mathbb{N}}$, again denoted by $\{(u_m^*, v_m^*)\}_{m \in \mathbb{N}}$, such that

$$(u_m^*(t, z), v_m^*(t, z)) \to (u^\diamond(t, z), v^\diamond(t, z)) \quad \text{as} \ \ m \to \infty \ \ \text{in} \ \ C_{loc}^{1,2}(\mathbb{R}, \mathbb{R}^2).$$

Moreover, $(u^\diamond(t,z), v^\diamond(t,z))$ satisfies $(u^\diamond(t,z), v^\diamond(t,z)) = (u^\diamond(t+T,z), v^\diamond(t+T,z))$ for all $(t,z) \in \mathbb{R}$, $\Lambda_1 \le u^\diamond(t,z) \le 1$ for all $(t,z) \in \mathbb{R}$, $v^\diamond(0,0) = \frac{1}{2}\Lambda_2$, $v^\diamond(0,z) \le \frac{1}{2}\Lambda_2$ for $z < 0$, and

$$\begin{cases} \partial_t u^\diamond = d_S u^\diamond_{zz}(t,z) - c u^\diamond_z(t,z) + \mu(t) - \beta(t)\frac{u^\diamond(t,z)v^\diamond(t,z)}{u^\diamond(t,z)+v^\diamond(t,z)} - \mu(t)u^\diamond(t,z), \\ \partial_t v^\diamond = v^\diamond_{zz}(t,z) - c v^\diamond_z(t,z) + \beta(t)\frac{u^\diamond(t,z)v^\diamond(t,z)}{u^\diamond(t,z)+v^\diamond(t,z)} - r(t)v^\diamond(t,z). \end{cases} \tag{3.18}$$

Similarly, to the proof of Theorem 3.8, we can obtain that

$$\inf_{t\in\mathbb{R},x\in\mathbb{R}} u^\diamond(t,z) \ge \Lambda_1, \qquad \liminf_{z\to+\infty} v^\diamond(t,z) > \epsilon \tag{3.19}$$

uniformly for $t \in \mathbb{R}$ for some constant $\epsilon > 0$.

It still has to show that $\lim_{z\to-\infty} u^\diamond(t,z) = 1$ and $\lim_{z\to-\infty} v^\diamond(t,z) = 0$. We first prove the latter. Here we use an argument similar to that in [36, Theorem 3.9]. On the contrary, we suppose that $\limsup_{z\to-\infty} \max_{t\in\mathbb{R}} v^\diamond(t,z) = \Lambda_3 \in \left(0, \frac{\Lambda_2}{2}\right)$. Then there exists a sequence $(t_j, z_j) \in [0,T) \times \mathbb{R}$ satisfying $t_j \to t^* \in [0,T]$ and $z_j \to -\infty$ as $j \to +\infty$ such that $\limsup_{j\to\infty} v^\diamond(t_j,z_j) = \Lambda_3$. Set $v_j(t,z) = v^\diamond(t, z+z_j)$ for any $(t,z) \in \mathbb{R}^2$. Then there exists a function $\Theta(t,z)$ such that $v_j(t,z) \to \Theta(t,z)$ in $C^{1,2}_{loc}(\mathbb{R}\times\mathbb{R})$ up to a subsequence. Moreover, $\Theta(t,z)$ satisfies $\Theta(t+T,z) = \Theta(t,z)$, $0 < \Theta(t,z) \le \Theta(t^*,0) = \Lambda_3$ for $(t,z) \in \mathbb{R}^2$. Due to (3.19), we have that $\Pi(t,x) = \Theta(t, x+c^*t)$ satisfies

$$\partial_t \Pi(t,x) \ge \partial_{xx}\Pi(t,x) + \left(\frac{\beta(t)\Lambda_1}{\Lambda_1 + \Pi(t,x)} - r(t)\right)\Pi(t,x), \quad (t,x) \in \mathbb{R}^2. \tag{3.20}$$

Let $U(t,x;\Pi(0,\cdot))$ be the solution of (3.20) with initial value $U(0,\cdot) = \Pi(0,\cdot)$. Then by the theory of the spreading speed for periodic evolution systems [19,20], we have that

$$\lim_{t\to\infty} \sup_{x\le(c^*-\varepsilon)t} \left|U(t,x;\Pi(0,\cdot)) - u^\triangle(t)\right| = 0$$

for any $\varepsilon \in (0,c^*)$. Consequently, by the comparison principle we can reach a contradiction, namely

$$\Lambda_3 \ge \Theta(nT, c^*nT) = \Pi(nT,0) \ge U(nT,0;\Pi(0,\cdot)) \to u^\triangle(nT) > \Lambda_3 \quad \text{as } n \to \infty.$$

Finally, we show $\lim_{z\to-\infty} u^\diamond(t,z) = 1$. Note that one already has $\lim_{z\to-\infty} v^\diamond(t,z) = 0$. Suppose on the contrary that $\lim_{z\to-\infty} \inf_{t\in\mathbb{R}} u^\diamond(t,z) = \Lambda_4 < 1$. Similar to the above argument, we have a function $\Upsilon(t,z) \in C^{1,2}(\mathbb{R}^2)$ such that $\Upsilon(t,z) = \Upsilon(t+T,z)$ and $\Lambda_4 = \Upsilon(t',0) \le \Upsilon(t,z) \le 1$ for any $(t,z) \in \mathbb{R}^2$. In particular, $\Upsilon(t,z)$ satisfies

$$\partial_t \Upsilon(t,z) = d_S \partial_{zz}\Upsilon(t,z) - c^*\partial_z \Upsilon(t,z) + \mu(t)(1 - \Upsilon(t,z)), \quad \forall (t,z) \in \mathbb{R}^2.$$

Let $\varphi(t, \frac{\Lambda_4}{2})$ be the solution of the following ODE with $\varphi(0) = \frac{\Lambda_4}{2}$

$$\varphi'(t) = \mu(t)(1 - \varphi(t)), \quad t \in \mathbb{R}.$$

Then we can also get a contradiction by using the comparison principle:

$$\Lambda_4 = \Upsilon(t',0) = \Upsilon(t'+nT,0) \ge \varphi(t'+nT, \frac{\Lambda_4}{2}) \to 1 > \Lambda_4$$

as $n \to \infty$. This completes the proof. □

Theorem 3.10. *Assume that $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} > 1$ and $0 < c < c^*$. Then system (1.1) admits no positive T-periodic traveling waves (u,v) satisfying $v(t,-\infty) = 0$ uniformly for $t \in \mathbb{R}$.*

Proof. Suppose, by contradiction, that for some $c_0 \in (0,c^*)$, system (1.1) has a periodic traveling wave $(u(t,x+c_0t), v(t,x+c_0t))$ such that $v(t,-\infty) = 0$ uniformly for $t \in \mathbb{R}$. Fix $\tilde{c} \in (c_0,c^*)$. Since $v(0,0) > 0$, it follows from Theorem 2.5 that there exists a constant $\epsilon > 0$ such that $\liminf_{t\to\infty} v(t, x+c_0t-\tilde{c}t) \ge \epsilon$ for each $x \in \mathbb{R}$. Then by the periodicity of $v(t,\cdot)$, we have

$$0 = \lim_{n\to\infty} v(0,(c_0-\tilde{c})nT) = \lim_{n\to\infty} v(nT,(c_0-\tilde{c})nT) \ge \epsilon > 0.$$

This is a contradiction. The proof is complete. □

Theorem 3.11. *Assume that* $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T r(t)dt} \leqslant 1$. *Then for any* $c \geqslant 0$, *system (1.1) admits no positive bounded* T*-periodic traveling waves* (u, v).

Proof. Suppose on the contrary that for some $c \geqslant 0$, system (1.1) admits a positive bounded T-periodic traveling waves (u, v). Then there exists a positive constant $\Lambda_* > 0$ such that $u(t, z) \leqslant \Lambda_*$ for $(t, z) \in \mathbb{R}^2$. By considering equation

$$\partial_t \hat{U} = \partial_{zz} \hat{U} - c\partial_z \hat{U} + \left(\frac{\beta(t)\Lambda_*}{\Lambda_* + \hat{U}(t, z)} - r(t) \right) \hat{U}(t, z), \quad t > 0,\, z \in \mathbb{R}$$

and using the comparison principle, we can obtain $v(t, z) \equiv 0$ on $(t, z) \in \mathbb{R}^2$ by similar arguments to those in [32, Theorem 3.1]. Clearly, there is a contradiction due to the fact that $v(t, z)$ is positive. This completes the proof. □

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