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ORIGINAL PAPER

# Existence of Ground State Sign-Changing Solutions of Fractional Kirchhoff-Type Equation with Critical Growth

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**Abstract**
In this paper, we study the following fractional Kirchhoff-type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^2K(x-y)dxdy)\mathcal{L}_K u=|u|^{2^*_\alpha-2}u+\mu f(u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\backslash\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary, $\alpha\in(0,1)$, $2\alpha<N<4\alpha$, $2^*_\alpha$ is the fractional critical Sobolev exponent and $\mu, a, b>0$; $\mathcal{L}_K$ is non-local integrodifferential operator. Under suitable conditions on $f$, for $\mu$ large enough, by using constraint variational method and the quantitative deformation lemma, we obtain a ground state sign-changing (or nodal) solution to this problem, and its energy is strictly larger than twice that of the ground state solutions.

**Keywords** Sign-changing solution · Non-local integrodifferential operator · Variational methods

## 1 Introduction and Main Results

In this article, we are interested in the existence of the ground state sign-changing solution for the following fractional Kirchhoff-type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u(y)|^2K(x-y)dxdy)\mathcal{L}_K u=|u|^{2^*_\alpha-2}u+\mu f(u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary, $\alpha \in (0, 1)$, $2\alpha < N < 4\alpha$, $2^*_\alpha = \frac{2N}{N-2\alpha}$ is the fractional critical Sobolev exponent and $\mu, a, b > 0$. Non-local integrodifferential operator $\mathcal{L}_K$ is defined by

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \backslash \{0\} \to (0, \infty)$ satisfies following properties:

$(K_1)$ $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{|x|^2, 1\}$;
$(K_2)$ there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-(N+2\alpha)}$ for any $x \in \mathbb{R}^N \backslash \{0\}$;
$(K_3)$ $K(-x) = K(x)$ for any $x \in \mathbb{R}^N \backslash \{0\}$.

As for the function $f$, we assume $f \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies the following hypotheses:

$(f_1)$ $\lim_{t\to 0} \frac{f(t)}{t^3} = 0$;
$(f_2)$ There exist $q \in (4, 2^*_\alpha)$ and $C > 0$ such that

$$|f(t)| \leq C(1 + |t|^{q-1}), \text{ for all } t \in \mathbb{R};$$

$(f_3)$ $\frac{f(t)}{|t|^3}$ is increasing on $\mathbb{R}\backslash\{0\}$.

***Remark 1.1*** (i) According to definition of $2^*_\alpha$ and condition $(f_2)$, it must need that $2^*_\alpha > 4$. So, in this paper, we must need $2\alpha < N < 4\alpha$. Certainly, we choice $\alpha \in (0, 1)$ and $\alpha$ is large so that $2\alpha < N < 4\alpha$ can satisfy.
(ii) If $f = |t|^{q-2}t$ for $q \in (4, 2^*_\alpha)$, then conditions $(f_1) - (f_3)$ are all satisfied.

The motivation to study problem (1.1) comes from the following general fractional Kirchhoff-type equation

$$\begin{cases} -(a + b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)dxdy)\mathcal{L}_K u = g(u), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.2}$$

When $K(x) = |x|^{-(N+2\alpha)}$, $\mathcal{L}_K$ is the fractional Laplace operator $(-\triangle)^\alpha$, which may be defined as

$$(-\triangle)^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|\, y \,|^{N+2\alpha}} dy,$$

and problem (1.2) reduces to the following fractional Kirchhoff equation

$$\begin{cases} (a + b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy)(-\triangle)^\alpha u = g(u), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.3}$$

It follows from Proposition 4.4 in [17] that $\lim_{\alpha\to 1^-}(-\triangle)^\alpha u = -\triangle u$. So, if $\alpha \to 1^-$, problem (1.3) becomes the following Kirchhoff equation

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = g(u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.4}$$

In the past decades, a great attention has been given to the Kirchhoff equation (1.4). Problem (1.4) is related to the following stationary analogue of the equation of Kirchhoff type

$$u_{tt}-(a+b\int_\Omega |\nabla u|^2dx)\Delta u = g(u), \tag{1.5}$$

which was introduced by Kirchhoff [24] as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-(\frac{P_0}{h}+\frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2dx)\frac{\partial^2 u}{\partial x^2} = g(u), \tag{1.6}$$

for free vibration of elastic strings.

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations, so the nonlocal term appears. For more mathematical and physical background on Kirchhoff type problems, we refer the readers to [6,38].

Because the so-called nonlocal term $(\int_\Omega |\nabla u|^2dx)\Delta u$ is involved in the equation, problem (1.4) is called nonlocal problem. The appearance of nonlocal term in the equations not only make its importance in many physical applications but also causes some difficulties and challenges from a mathematical point of view. This fact makes the study of problem (1.4) or similar problems particularly interesting. After the pioneer work of Lions [27], in which a functional analysis approach was proposed to Eq. (1.5) with Dirichlet boundary condition, a lots of interesting results to problem (1.4) or similar Kirchhoff-type equations were obtained in last decades, for the sake of space, we do not list them here.

Especially, there are some interesting results about sign-changing solutions to problem (1.4) or similar Kirchhoff-type equations. For examples, Zhang and Perera [56], and Mao and Zhang [33] used the method of invariant sets of descent flow [28] to obtain the existence of sign-changing solution for problem (1.4). In [19], Figueiredo and Nascimento considered the following Kirchhoff equation of the type

$$\begin{cases} -M(\int_\Omega |\nabla u|^2dx)\Delta u = f(u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.7}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^3$, $M$ is a general $C^1$ class function and $f$ is a superlinear $C^1$ class function with subcritical growth. By using minimization argument and a quantitative deformation lemma, the existence of a sign-changing solution for this Kirchhoff equation was obtained.

In unbounded domains, Figueiredo and Santos Júnior [20] studied a class of nonlocal Schrödinger–Kirchhof problems involving only continuous functions. Using a minimization argument and a quantitative deformation lemma, they obtained a least energy sign-changing solution to Schrödinger–Kirchhoff problems. Moreover, when the problem presents symmetry, the authors showed that it has infinitely many nontrivial solutions.

It is noticed that, combining constraint variational methods and quantitative deformation lemma, Shuai [40] studied the existence and asymptotic behavior of least energy sign-changing solution to problem (1.4). Latter, under some weaker assumptions on $f$ (especially, Nehari type monotonicity condition has been removed), with the aid of some new analytical skills and Non-Nehari manifold method, Tang and Cheng [43] improved and generalized some results obtained in [40].

In [14], Deng et al. studied the following Kirchhoff problem

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u=f(x,u),\ \text{ in } \mathbb{R}^3. \tag{1.8}$$

The authors proved the existence of a sign-changing solution to Kirchhoff problem (1.8), which changes signs exactly $k$ times for any $k\in\mathbb{N}$. Moreover, they investigated the energy property and the asymptotic behavior of sign-changing solution.

By using invariant sets method of descent flow or the Ljusternik–Schnirelman type minimax method, in [7,42], the existence of multiple sign-changing solutions for some Kirchhoff problem were considered.

For more results on sign-changing solutions for Kirchhoff-type equations, we refer the reader to [12,26,29,32,39,49,55] and the references therein.

On the other hand, when $a=1$, $b=0$, problem (1.2) and problem (1.3) stem from the following problems

$$\begin{cases} -\mathcal{L}_K u=g(x,u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.9}$$

and

$$\begin{cases} (-\triangle)^\alpha u=g(x,u), & x\in\Omega,\\ u=0, & x\in\mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.10}$$

respectively.

Different from the operator $-\triangle$, the fractional Laplacian operator $(-\triangle)^\alpha$ is nonlocal. From a physical point of view, nonlocal operators play a crucial rule in describing several different physical phenomena, details are available for Refs. [1,3,18,25,34,37,45]. It is noticed that, to overcome some difficulties that come from the nonlocal feature of fractional Laplacian, Caffarelli and Silvestre developed a powerful extension method in [5]. By using the extension method, one can transform Eq. (1.10) into a local problem settled on $\mathbb{R}^N_+$. Because there has nonlocal operator $(-\triangle)^\alpha$, from a mathematical point of view, the existence of sign-changing solution is an interesting and important aspect to problem (1.10). By applying the Caffarelli–Silvestre extension method and invariant sets of descending flow, Chang and Wang [8] obtained the existence and multiplicity of sign-changing solution to problem (1.10). In [13], via variational method combining invariant sets of descending, Deng and Shuai proved that problem (1.10) has a positive solution, a negative solution and a sign-changing solution under suitable conditions; meanwhile, when $g(x, u)$ satisfies a monotonicity condition, they showed that the problem has a ground state sign-changing solution with its energy is strictly larger than that of the ground state solution of Nehari type; Moreover, if $g(x, u)$ is odd in $u$, they obtained an unbounded sequence of sign-changing solutions by using genus and relative genus. For more results on sign-changing solutions for problem (1.10) or similar fractional Laplacian equations, we refer the reader to [4,22,44,53] and the reference therein.

However, it's worth noting that we do not know whether the Caffarelli–Silvestre extension method is still valid for the general non-local integrodifferential operator $\mathcal{L}_K$ or not. It seems that, to the best of our knowledge, problem (1.9) was first studied by Servadei and Valdinoci [36]. In fact, Servadei and Valdinoci established the variational setting and obtained a nontrivial solution of problem (1.9) via mountain pass theorem. Fortunately, many of additional difficulties coming from nonlocal feature of the nonlocal integrodifferential operator have been overcame by Servadei and Valdinoci.

Regarding the existence of sign-changing solutions for problem (1.9), there are few results in the literature. In [23], Gu et al. obtained the existence of infinitely many sign-changing solutions to problem (1.9) by combining critical point theory and invariant sets of descending flow. When $g(x, u)$ is asymptotically linear at infinity with respect to $u$, Luo et al. [31] investigated the existence of sign-changing solutions for problem (1.9).

The nonlocal term in problem (1.2) and problem (1.3), which can be regarded as a combination of the general non-local integrodifferential operator $\mathcal{L}_K$ or the fractional Laplacian operator $(-\triangle)^\alpha$ and the nonlocal term of Kirchhoff type, makes problem (1.2) and problem (1.3) even more complicated. Due to this fact, comparing with problem (1.9) and problem (1.10) respectively, there are very few results involving sign-changing solutions for problem (1.2) and problem (1.3). In [11], Cheng and Gao considered following fractional Kirchhoff equation

$$\Big(a + b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy\Big)(-\triangle)^\alpha u + V(x)u = g(x, u), \quad x \in \mathbb{R}^N, \tag{1.11}$$

where $\alpha \in (0, 1)$ and $2\alpha < N$, $a$ and $b$ are positive constant. By the minimization argument on the nodal Nehari manifold and quantitative deformation lemma, they proved the existence and asymptotic behavior of ground state sign-changing solutions for problem (1.11). Latterly, without the usual Nehari-type monotonicity condition on $g$, Chen et al. [10] improved and generalized some results obtained in [11].

Very recently, by using the non-Nehari manifold method, Luo et al. [30] studied the existence of ground state sign-changing solutions to fractional Kirchhoff equation (1.2). Their results generalized some results obtained by [40,43].

However, regarding the existence of sign-changing solutions for Kirchhoff-type equations, to the best of our knowledge, it seems that few results involved the sign-changing solutions to critical problem [46,51]. On the other hand, there are some essential differences in studying the sign-changing solutions for Kirchhoff-type equations between critical problem and subcritical problem. Obviously, essential differences come from the critical term. So, from a mathematical point of view, the problem of sign-changing solutions for Kirchhoff-type equations with critical growth is challenged and interested. Inspired by above results, in this paper, we investigate the existence and asymptotic behavior of ground state sign-changing solutions for critical problem (1.1). It is noticed that there are some interesting results, for example [2,9,41,47,48,50,52,57], considered sign-changing solutions for Schrödinger–Poisson systems.

Before presenting our main results, we denote by $L^p(\Omega)$ a Lebesgue space with the norm $|u|_p := (\int_\Omega |u|^p dx)^{\frac{1}{p}}$, $1 \leq p < \infty$.

Denote $H^\alpha(\Omega)$ the usual fractional Sobolev space equipped with the inner product and norm

$$
\begin{aligned}
(u, v)_{H^\alpha(\Omega)} &= \int_\Omega uvdx + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{| x - y |^{N+2\alpha}} dxdy, \\
\|u\|_{H^\alpha(\Omega)} &= (u, u)^{\frac{1}{2}}_{H^\alpha(\Omega)} = (\int_\Omega |u|^2 dx + \int_\Omega \int_\Omega \frac{| u(x) - u(y) |^2}{| x - y |^{N+2\alpha}} dxdy)^{\frac{1}{2}}.
\end{aligned} \tag{1.12}
$$

Let

$$
\begin{aligned}
X =& \{u : \mathbb{R}^N \to \mathbb{R} | u \text{ is Lebesgue measurable, } u|_\Omega \in L^2(\Omega) \\
& \text{and } \int_Q |u(x) - u(y)|^2 K(x - y) dxdy < \infty\},
\end{aligned} \tag{1.13}
$$

where $Q = \mathbb{R}^{2N} \backslash (\Omega^c \times \Omega^c)$ and $\Omega^c = \mathbb{R}^N \backslash \Omega$. The function space $X$ is equipped with the following norm:

$$
\|u\|_X^2 = \| u \|_2^2 + \int_Q |u(x) - u(y)|^2 K(x - y) dxdy.
$$

It is noticed that the space $X$ was introduced by Servadei and Valdinoci [36].

Set

$$
X_0 = \{u \in X : u = 0 \text{ a.e. } \mathbb{R}^N \backslash \Omega\}
$$

with norm

$$
\|u\|^2 = \int_Q |u(x) - u(y)|^2 K(x - y) dxdy, \tag{1.14}
$$

and scalar product

$$(u, v) := \int_Q (u(x) - u(y))(v(x) - v(y))K(x - y)dxdy, \forall u, v \in X_0.$$

We should notice that the norms in (1.12) and (1.13) are not the same (even in the case of $K(x) = |x|^{-(N+2\alpha)}$), because $\Omega \times \Omega$ is strictly contained in $Q$. This fact implies that the usual fractional Sobolev space approach is not sufficient for studying problem (1.1).

The energy functional associated with problem (1.1) is defined by

$$\begin{aligned} I(u) &= \frac{a}{2}\int_Q |u(x) - u(y)|^2 K(x - y)dxdy + \frac{b}{4}\left(\int_Q |u(x) - u(y)|^2 K(x - y)dxdy\right)^2 \\ &\quad - \mu \int_\Omega F(u)dx - \frac{1}{2_\alpha^*}\int_\Omega |u|^{2_\alpha^*}dx \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \mu\int_\Omega F(u)dx - \frac{1}{2_\alpha^*}\int_\Omega |u|^{2_\alpha^*}dx, \end{aligned}$$

for any $u \in X_0$.

Moreover, under our conditions, $I(u)$ belongs to $C^1$, and the Fréchet derivative of $I$ is

$$\langle I'(u), v\rangle = (a + b\|u\|^2)(u, v) - \mu\int_\Omega f(u)vdx - \int_\Omega |u|^{2_\alpha^*-2}uvdx$$

for any $u, v \in X_0$.

The solution of problem (1.1) is the critical point of the functional $I(u)$. Furthermore, if $u \in X_0$ is a solution of problem (1.1) and $u^\pm \neq 0$, then $u$ is a sign-changing solution of problem (1.1), where

$$u^+ = \max\{u(x), 0\}, \quad u^- = \min\{u(x), 0\}.$$

Our goal in this paper is to seek the ground state sign-changing solutions to problem (1.1). So, we borrow some ideals from [10,11,19,20,22,26,30,40,43,46,51,55]. That is, we first try to seek a minimizer of the energy functional $I$ over the following constraint:

$$\mathcal{M} = \{u \in X_0, u^\pm \neq 0 \text{ and } \langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0\}, \tag{1.15}$$

and then prove that the minimizer is a sign-changing solution of problem (1.1).

However, since the lack of the compactness caused by the critical term, it is rather difficult to show that $\inf_{u\in\mathcal{M}} I(u)$ is achieved in $\mathcal{M}$. This problem prevent us from using the standard way. So, we need some new ideas to deal with this essential problem (see Lemmas 2.3 and 2.4).

The main results can be stated as follows.

**Theorem 1.1** *Suppose that* $(f_1)-(f_3)$ *are satisfied. Then, there exists* $\mu^\star > 0$ *such that for all* $\mu \geq \mu^\star$*, the problem* (1.1) *has a ground state sign-changing solution* $u$*.*

We first prove several technical lemmas, especially Lemmas 2.3 and 2.4, and obtain that $c^\mu = \inf_{u\in\mathcal{M}} I(u)$ is achieved in $\mathcal{M}$ for $\mu$ large enough, and then we prove that the minimizer of $c^\mu$ is just a least energy sign-changing solution of (1.1).

**Theorem 1.2** *Suppose that* $(f_1)$–$(f_3)$ *are satisfied. Then, there exists* $\mu^{\star\star} > 0$ *such that for all* $\mu \geq \mu^{\star\star}$*, the* $c^* > 0$ *is achieved and*

$$I(u) > 2c^*,$$

*where* $c^* = \inf_{u\in\mathcal{N}} I(u)$*,* $\mathcal{N} = \{u \in X_0\backslash \{0\}|\langle I'(u), u\rangle = 0\}$*, and* $u$ *is the ground state sign-changing solution obtained in Theorem 1.1. In particular,* $c^* > 0$ *is achieved either by a positive or a negative function.*

Theorem 1.2 indicates that the energy of each sign-changing solution of (1.1) is strictly larger than two times of the ground state energy, i.e., the least energy among all solutions in $X_0$ of (1.1).

The paper is organized as follows. In Sect. 2, we prove several technical lemmas, especially Lemmas 2.3 and 2.4, which are crucial to prove our main results. In Sect. 3, by quantitative deformation lemma and degree theory, we first obtain that the minimizer of the constrained problem is a sign-changing solution. Then, by energy comparisons, we prove Theorem 1.2.

## 2 Technical Lemmas

In this section, we prove several technical lemmas, which help us use constraint minimization on $\mathcal{M}$ to seek a critical point of $I$. Firstly, we cit one important imbedding theorem about fractional Sobolev space.

**Lemma 2.1** ([36]) *Let* $\alpha \in (0, 1), 2\alpha < N$*,* $\Omega$ *be an open bounded set of* $\mathbb{R}^N$*, then the embedding* $X_0 \hookrightarrow L^r(\Omega)$ *is continuous for any* $r \in [2, 2^*_\alpha]$ *and compact for any* $r \in [2, 2^*_\alpha)$*.*

Now, fixed $u \in X_0$ with $u^\pm \neq 0$, define function $\psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $W_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\begin{aligned}
\psi_u(s,t) &= I(su^+ + tu^-),\\
W_u(s,t) &= (\langle I'(su^+ + tu^-), su^+\rangle, \langle I'(su^+ + tu^-), tu^-\rangle).
\end{aligned}$$

The following lemma shows that the set $\mathcal{M}$ is nonempty in $X_0$.

**Lemma 2.2** *Assume that* $(f_1)$–$(f_3)$ *hold, if* $u \in X_0$ *with* $u^\pm \neq 0$*, then* $\psi_u$ *has the following properties:*

(i) *The pair* $(s,t)$ *is a critical point of* $\psi_u$ *with* $s, t > 0$ *if and only if* $su^+ + tu^- \in \mathcal{M}$*, where* $\mathcal{M}$ *is defined as in* (1.15)*;*

(ii) *The function* $\psi_u$ *has a unique critical point* $(s_u, t_u)$ *on* $(0,\infty)\times(0,\infty)$*, which is also the unique maximum point of* $\psi_u$ *on* $[0,\infty)\times[0,\infty)$*; Furthermore, if* $\langle I'(u), u^+\rangle \le 0$ *and* $\langle I'(u), u^-\rangle \le 0$ *then* $0 < s_u, t_u \le 1$.

***Proof*** Let

$$\Theta(u) := -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u^-(x)u^+(y)+u^-(y)u^+(x))K(x-y)dxdy.$$

(i) By definition of $\psi_u$, we have that

$$\begin{aligned}\nabla\psi_u(s,t) &= (\langle I'(su^+ + tu^-), u^+\rangle, \langle I'(su^+ + tu^-), u^-\rangle)\\ &= (\frac{1}{s}\langle I'(su^+ + tu^-), su^+\rangle, \frac{1}{t}\langle I'(su^+ + tu^-), \beta u^-\rangle),\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\langle I'(su^+ + tu^-), su^+\rangle &= as^2\|u^+\|^2 + bs^4\|u^+\|^4 + bs^2t^2\|u^+\|^2\|u^-\|^2 + ast\Theta(u)\\ &+ 3bs^3t\|u^+\|^2\Theta(u) + bst^3\|u^-\|^2\Theta(u) + 2bs^2t^2\Theta^2(u)\\ &- s^{2^*_\alpha}\int_\Omega |u^+|^{2^*_\alpha}dx - \mu\int_\Omega f(su^+)su^+dx,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\langle I'(su^+ + tu^-), tu^-\rangle &= at^2\|u^-\|^2 + bt^4\|u^-\|^4 + bs^2t^2\|u^+\|^2\|u^-\|^2 + ast\Theta(u)\\ &+ 3bst^3\|u^-\|^2\Theta(u) + bs^3t\|u^+\|^2\Theta(u) + 2bs^2t^2\Theta^2(u)\\ &- t^{2^*_\alpha}\int_\Omega |u^-|^{2^*_\alpha}dx - \mu\int_\Omega f(tu^-)tu^-dx.\end{aligned} \tag{2.3}$$

From (2.1), the definition of $\mathcal{M}$ and facts $(su^+ + tu^-)^+ = su^+$ and $(su^+ + tu^-)^- = tu^-$, item *(i)* is obvious.

(ii) Firstly, we prove the existence of $s_u$ and $t_u$.

From $(f_1)$ and $(f_2)$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ satisfies

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^{q-1}, \tag{2.4}$$

for all $t \in \mathbb{R}$.

On the other hand, by $(f_1)$ and $(f_3)$, we have that

$$f(t)t > 0, t \neq 0;\ \ F(t) \ge 0, t \in \mathbb{R}. \tag{2.5}$$

Then, by (2.4), (2.5) and Sobolev embedding theorem, there exist $\tau_1 > 0$ and $\tau_2$ large enough such that

$$\langle I'(\tau_1 u^+ + tu^-), \tau_1 u^+\rangle > 0, \langle I'(su^+ + \tau_1 u^-), \tau_1 u^-\rangle > 0 \tag{2.6}$$

for all $s, t \geq 0$;

$$\langle I'(\tau_2 u^+ + t u^-), \tau_2 u^+ \rangle < 0, \langle I_b'(s u^+ + \tau_2 u^-), \tau_2 u^- \rangle < 0 \tag{2.7}$$

for all $s, t \in [\tau_1, \tau_2]$.

So, together (2.6), (2.7) with Miranda's Theorem [35], there exists $(s_u, t_u) \in (0, \infty) \times (0, \infty)$ such that $W_u(s, t) = (0, 0)$, i.e., $s_u u^+ + t_u u^- \in \mathcal{M}$.

In the following, we prove the uniqueness of the pair $(s_u, t_u)$. By standard arguments, we only prove the uniqueness in case of $u \in \mathcal{M}$ here.

For any $u \in \mathcal{M}$, we have that

$$\begin{aligned} a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 + (a + 3b\|u^+\|^2 + b\|u^-\|^2 + 2b\Theta(u))\Theta(u) \\ = \int_\Omega |u^+|^{2^*_\alpha} dx + \mu \int_\Omega f(u^+)u^+ dx, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} a\|u^-\|^2 + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 + (a + 3b\|u^-\|^2 + b\|u^+\|^2 + 2b\Theta(u))\Theta(u) \\ = \int_\Omega |u^-|^{2^*_\alpha} dx + \mu \int_\Omega f(u^-)u^- dx. \end{aligned} \tag{2.9}$$

Let $(s_0, t_0)$ be a pair of numbers such that $s_0 u^+ + t_0 u^- \in \mathcal{M}$ with $0 < s_0 \leq t_0$. So, one has that

$$\begin{aligned} & a s_0^2\|u^+\|^2 + b s_0^4\|u^+\|^4 + b s_0^2 t_0^2\|u^+\|^2\|u^-\|^2 \\ & \quad + (a s_0 t_0 + 3b s_0^3 t_0\|u^+\|^2 + b s_0 t_0^3\|u^-\|^2 + 2b s_0^2 t_0^2\Theta(u))\Theta(u) \\ & = s_0^{2^*_\alpha} \int_\Omega |u^+|^{2^*_\alpha} dx + \mu \int_\Omega f(s_0^2 u^+) s_0^2 u^+ dx, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & a t_0^2\|u^-\|^2 + b t_0^4\|u^-\|^4 + b s_0^2 t_0^2\|u^+\|^2\|u^-\|^2 \\ & \quad + (a s_0 t_0 + 3b t_0^3 s_0\|u^-\|^2 + b t_0 s_0^3\|u^+\|^2 + 2b s_0^2 t_0^2\Theta(u))\Theta(u) \\ & = t_0^{2^*_\alpha} \int_\Omega |u^-|^{2^*_\alpha} dx + \mu \int_\Omega f(t_0 u^-) t_0 u^- dx. \end{aligned} \tag{2.11}$$

Hence, thanks to $0 < s_0 \leq t_0$, we have that

$$\begin{aligned} & \frac{a\|u^-\|^2}{t_0^2} + \frac{a\Theta(u)}{t_0^2} + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 + (3b\|u^-\|^2 + b\|u^+\|^2 + 2b\Theta(u))\Theta(u) \\ & \geq t_0^{2^*_\alpha - 4} \int_\Omega |u^-|^{2^*_\alpha} dx + \mu \int_\Omega \left[\frac{f(t_0 u^-)}{(t_0 u^-)^3}\right] (u^-)^4 dx. \end{aligned} \tag{2.12}$$

Combining (2.9) with (2.12), one has that

$$a\left(\frac{1}{t_0^2}-1\right)(\|u^-\|^2+\Theta(u)) \geq (t_0^{2_\alpha^*-4}-1)\int_\Omega |u^-|^{2_\alpha^*}dx$$
$$+\mu\int_\Omega\left[\frac{f(t_0u^-)}{(t_0u^-)^3}-\frac{f(u^-)}{(u^-)^3}\right](u^-)^4dx.$$

If $t_0>1$, the left side of above inequality is negative, which is absurd because the right hand side is positive by condition $(f_3)$. Therefore, we obtain that $0<s_0\leq t_0\leq 1$.

Similarly, by (2.10) and $0<s_0\leq t_0$, we get

$$a\left(\frac{1}{s_0^2}-1\right)(\|u^+\|^2+\Theta(u)) \leq (s_0^{2_\alpha^*-4}-1)\int_\Omega |u^+|^{2_\alpha^*}dx$$
$$+\mu\int_\Omega\left[\frac{f(s_0u^+)}{(s_0u^+)^3}-\frac{f(u^+)}{(u^+)^3}\right](u^+)^4dx.$$

In view of $(f_3)$, we have that $s_0\geq 1$. Consequently, $s_0=t_0=1$.

So, we conclude that $(s_u,t_u)=(1,1)$ is the unique pair of numbers such that $s_uu^++t_uu^-\in\mathcal{M}$ if $u\in\mathcal{M}$.

Next, we will prove that $(s_u,t_u)$ is the unique maximum point of $\psi_u$ on $[0,\infty)\times[0,\infty)$.

For any $u\in X_0$ with $u^\pm\neq 0$, by (2.5), we have that

$$\lim_{|(s,t)|\to\infty}\psi_u(s,t)=-\infty.$$

Hence, $(s_u,t_u)$ is the unique critical point of $\psi_u$ in $[0,\infty)\times[0,\infty)$. On the other hand, let $s_0,t_0\geq 0$ be fixed, it is easy to see that $\psi_u(s,t_0)$ and $\psi_u(s_0,t)$ are increasing functions if $s$ and $t$ is small enough respectively. So, maximum point of $\psi_u$ cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$.

Lastly, we will prove that $0<s_u,t_u\leq 1$ if $\langle I'(u),u^\pm\rangle\leq 0$.

Suppose $s_u\geq t_u>0$. By $s_uu^++t_uu^-\in\mathcal{M}$, one has

$$\begin{aligned}
&as_u^2\|u^+\|^2+bs_u^4\|u^+\|^4+bs_u^4\|u^+\|^2\|u^-\|^2\\
&\quad+(as_u^2+3bs_u^4\|u^+\|^2+bs_u^4\|u^-\|^2+2bs_u^4\Theta(u))\Theta(u)\\
&\geq as_u^2\|u^+\|^2+bs_u^4\|u^+\|^4+bs_u^2t_u^2\|u^+\|^2\|u^-\|^2\\
&\quad+(as_ut_u+3bs_u^3t_u\|u^+\|^2+bs_ut_u^3\|u^-\|^2+2bs_u^2t_u^2\Theta(u))\Theta(u)\\
&=s_u^{2_\alpha^*}\int_\Omega|u^+|^{2_\alpha^*}dx+\mu\int_\Omega f(s_u^2u^+)s_u^2u^+dx.
\end{aligned}\tag{2.13}$$

On the other hand, by $\langle I'(u),u^+\rangle\leq 0$, one has

$$a\|u^+\|^2+b\|u^+\|^4+b\|u^+\|^2\|u^-\|^2+(a+3b\|u^+\|^2+b\|u^-\|^2+2b\Theta(u))\Theta(u)$$

$$\leq \int_\Omega |u^+|^{2^*_\alpha} dx + \mu \int_\Omega f(u^+)u^+ dx. \tag{2.14}$$

According to (2.13) and (2.14), we have that

$$a(\frac{1}{s_u^2} - 1)(\|u^+\|^2 + \Theta(u)) \geq (s_u^{2^*_\alpha - 4} - 1) \int_\Omega |u^+|^{2^*_\alpha} dx$$
$$+ \mu \int_\Omega [\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}](u^+)^4 dx.$$

Thanks to condition $(f_3)$, we get $s_u \leq 1$. Thus, we have that $0 < s_u, t_u \leq 1$. □

The following result is very important to prove that $c^\mu = \inf_{u \in \mathcal{M}} I(u)$ is achieved.

**Lemma 2.3** *Let $c^\mu = \inf_{u \in \mathcal{M}} I(u)$, then we have that*

$$\lim_{\mu \to \infty} c^\mu = 0.$$

***Proof*** By (2.4) and Sobolev inequalities, we can prove that there exists $\rho > 0$ such that $\|u^\pm\| \geq \rho$ for all $u \in \mathcal{M}$.

For any $u \in \mathcal{M}$, it is obvious that $\langle I'(u), u \rangle = 0$. Thanks to $(f_3)$, we conclude that

$$f(t)t - 4F(t) \geq 0, \tag{2.15}$$

and is increasing when $t > 0$ and decreasing when $t < 0$.

Then, ones get

$$\begin{aligned} I(u) &= I(u) - \frac{1}{4}\langle I'(u), u \rangle \\ &= \frac{a}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{2^*_\alpha}) \int_\Omega |u|^{2^*_\alpha} dx + \frac{\mu}{4} \int_\Omega [f(u)u - 4F(u)] dx \\ &\geq \frac{a}{4}\|u\|^2, \end{aligned}$$

for any $u \in \mathcal{M}$.

So, $I$ is bounded below on $\mathcal{M}$, that is, $c^\mu = \inf_{u \in \mathcal{M}} I(u)$ is well-defined.

Let $u \in X_0$ with $u^\pm \neq 0$ be fixed. By Lemma 2.2, for each $\mu > 0$, there are $s_\mu, t_\mu > 0$ such that $s_\mu u^+ + t_\mu u^- \in \mathcal{M}$.

Thanks to (2.5), we have that

$$\begin{aligned} 0 \leq c^\mu &= \inf_{u \in \mathcal{M}} I(u) = I(s_\mu u^+ + t_\mu u^-) \\ &\leq \frac{a}{2}\|s_\mu u^+ + t_\mu u^-\|^2 + \frac{b}{4}\|s_\mu u^+ + t_\mu u^-\|^4 \\ &\leq a s_\mu^2 \|u^+\|^2 + a t_\mu^2 \|u^-\|^2 + 2b s_\mu^4 \|u^+\|^4 + 2b t_\mu^4 \|u^-\|^4. \end{aligned}$$

Next, we prove that $s_\mu \to 0$ and $t_\mu \to 0$, as $\mu \to \infty$.
Let

$$\Pi_u = \{(s_\mu, t_\mu) \in [0, \infty) \times [0, \infty) : W_u(s_\mu, t_\mu) = (0, 0), \mu > 0\},$$

where $W_u$ was defined as in Lemma 2.2. Thanks to (2.5), ones has

$$\begin{aligned}
&s_\mu^{2_\alpha^*} \int_\Omega |u^+|^{2_\alpha^*} dx + t_\mu^{2_\alpha^*} \int_\Omega |u^-|^{2_\alpha^*} dx \\
&\le s_\mu^{2_\alpha^*} \int_\Omega |u^+|^{2_\alpha^*} dx + t_\mu^{2_\alpha^*} \int_\Omega |u^-|^{2_\alpha^*} dx \\
&\quad + \mu \int_\Omega f(s_\mu u^+) s_\mu u^+ dx + \mu \int_\Omega f(t_\mu u^-) t_\mu u^- dx \\
&= a\|s_\mu u^+ + t_\mu u^-\|^2 + b\|s_\mu u^+ + t_\mu u^-\|^4 \\
&\le 2as_\mu^2 \|u^+\|^2 + 2at_\mu^2 \|u^-\|^2 + 8bs_\mu^4 \|u^+\|^4 + 8bt_\mu^4 \|u^-\|^4.
\end{aligned}$$

Since $2_\alpha^* > 4$, we conclude that $\Pi_u$ is bounded.

Let $\{\mu_n\} \subset (0, \infty)$ be such that $\mu_n \to \infty$ as $n \to \infty$. Then, in subsequence sense, there are $s_0$ and $t_0$ such that

$$(s_{\mu_n}, t_{\mu_n}) \to (s_0, t_0),$$

as $n \to \infty$.

We prove $s_0 = t_0 = 0$. By contradiction, we suppose that $s_0 > 0$ or $t_0 > 0$. Thanks to $s_{\mu_n} u^+ + t_{\mu_n} u^- \in \mathcal{M}$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&a\|s_{\mu_n} u^+ + t_{\mu_n} u^-\|^2 + b\|s_{\mu_n} u^+ + t_{\mu_n} u^-\|^4 \\
&\quad = \int_\Omega |s_{\mu_n} u^+ + t_{\mu_n} u^-|^{2_\alpha^*} dx + \mu_n \int_\Omega f(s_{\mu_n} u^+ + t_{\mu_n} u^-)(s_{\mu_n} u^+ + t_{\mu_n} u^-) dx.
\end{aligned} \tag{2.16}$$

According to $s_{\mu_n} u^+ \to s_0 u^+$ and $t_{\mu_n} u^- \to t_0 u^-$ in $X_0$, (2.4) and (2.5), we have that

$$\begin{aligned}
&\int_\Omega f(s_{\mu_n} u^+ + t_{\mu_n} u^-)(s_{\mu_n} u^+ + t_{\mu_n} u^-) dx \\
&\qquad \to \int_\Omega f(s_0 u^+ + t_0 u^-)(s_0 u^+ + t_0 u^-) dx > 0,
\end{aligned}$$

as $n \to \infty$. Then, we conclude a contradiction with the equality (2.16).

Hence, $s_0 = t_0 = 0$. That is, $\lim_{\mu \to \infty} c^\mu = 0$. □

With the above results, we shall proceed through three steps to obtain that $c^\mu = \inf_{u \in \mathcal{M}} I(u)$ is achieved.

**Lemma 2.4** *There exist $\mu^\star > 0$ such that for all $\mu \geq \mu^\star$, the infimum $c^\mu$ is achieved.*

***Proof*** According to definition of $c^\mu$, there is a sequence $\{u_n\} \subset \mathcal{M}$ such that

$$\lim_{n\to\infty} I(u_n) = c^\mu.$$

Obviously, $\{u_n\}$ is bounded in $X_0$. Then, up to a subsequence, still denoted by $\{u_n\}$, there exist $u \in X_0$ such that $u_n \rightharpoonup u$.

By Lemma 2.1, for all $p \in [2, 2_\alpha^*)$, we have

$$\begin{aligned} &u_n \to u \ \text{ in } L^p(\Omega),\\ &u_n(x) \to u(x) \ \text{ a.e. } x \in \Omega. \end{aligned}$$

So,

$$\begin{aligned} &u_n^\pm \rightharpoonup u^\pm \ \text{ in } X_0,\\ &u_n^\pm \to u^\pm \ \text{ in } L^p(\Omega),\\ &u_n^\pm(x) \to u^\pm(x) \ \text{ a.e. } x \in \Omega. \end{aligned}$$

Denote $\delta := \frac{\alpha}{N} S^{\frac{N}{2\alpha}}$, where

$$S := \inf_{u\in X_0\setminus\{0\}} \frac{\|u\|^2}{(\int_\Omega |u|^{2_\alpha^*} dx)^{\frac{2}{2_\alpha^*}}}.$$

According to Lemma 2.3, there is $\mu^\star > 0$ such that $c_b^\mu < \delta$ for all $\mu \geq \mu^\star$.

Fix $\mu \geq \mu^\star$, it follows from Lemma 2.2 that

$$I(su_n^+ + tu_n^-) \leq I(u_n)$$

for all $s, t \geq 0$.

By using Brezis–Lieb Lemma and Fatou's Lemma, we have that

$$\begin{aligned} &\liminf_{n\to\infty} I(su_n^+ + tu_n^-)\\ &\geq \frac{as^2}{2} \lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{at^2}{2} \lim_{n\to\infty} (\|u_n^- - u^-\|^2 + \|u^-\|^2)\\ &\quad + \frac{bs^4}{4} [\lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2)]^2 + \frac{bt^4}{4} [\lim_{n\to\infty} (\|u_n^- - u^-\|^2 + \|u^-\|^2)]^2\\ &\quad - \frac{s_u^{2_\alpha^*}}{2_\alpha^*} \lim_{n\to\infty} (|u_n^+ - u^+|_{2_\alpha^*}^{2_\alpha^*} + |u^+|_{2_\alpha^*}^{2_\alpha^*}) - \frac{t_u^{2_\alpha^*}}{2_\alpha^*} \lim_{n\to\infty} (|u_n^- - u^-|_{2_\alpha^*}^{2_\alpha^*} + |u^-|_{2_\alpha^*}^{2_\alpha^*})\\ &\quad - \mu \int_\Omega F(su^+)dx - \mu \int_\Omega F(tu^-)dx + ast \liminf_{n\to\infty} \Theta(u_n) \end{aligned}$$

$$
\begin{aligned}
&+ bs^2t^2 \liminf_{n\to\infty} \Theta^2(u_n) + \frac{bs^2t^2}{2} \liminf_{n\to\infty}(\|u_n^+\|^2\|u_n^-\|^2)\\
&+ bs^3t \liminf_{n\to\infty}(\|u_n^+\|^2\Theta(u_n)) + bt^3s \liminf_{n\to\infty}(\|u_n^-\|^2\Theta(u_n))\\
\geq\ & I(su^+ + tu^-) + \frac{as^2}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|^2 + \frac{at^2}{2} \lim_{n\to\infty} \|u_n^- - u^-\|^2\\
&+ \frac{bs^4}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|^2\|u^+\|^2 + \frac{bt^4}{2} \lim_{n\to\infty} \|u_n^- - u^-\|^2\|u^-\|^2\\
&+ \frac{bs^4}{4}(\lim_{n\to\infty} \|u_n^+ - u^+\|^2)^2 + \frac{bt^4}{4}(\lim_{n\to\infty} \|u_n^- - u^-\|^2)^2\\
&- \frac{s^{2_\alpha^*}}{2_\alpha^*} \lim_{n\to\infty} |u_n^+ - u^+|_{2_\alpha^*}^{2_\alpha^*} - \frac{t^{2_\alpha^*}}{2_\alpha^*} \lim_{n\to\infty} |u_n^- - u^-|_{2_\alpha^*}^{2_\alpha^*}\\
=\ & I(su^+ + tu^-) + \frac{as^2}{2}A_1 + \frac{bs^4}{2}A_1\|u^+\|^2 + \frac{bs^4}{4}A_1^2 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\\
&+ \frac{at^2}{2}A_2 + \frac{bt^4}{2}A_2\|u^-\|^2 + \frac{bt^4}{4}A_2^2 - \frac{t^{2_\alpha^*}}{2_\alpha^*}B_2
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \lim_{n\to\infty} \|u_n^+ - u^+\|^2,\ A_2 = \lim_{n\to\infty} \|u_n^- - u^-\|^2,\\
B_1 &= \lim_{n\to\infty} |u_n^+ - u^+|_{2_\alpha^*}^{2_\alpha^*},\ B_2 = \lim_{n\to\infty} |u_n^- - u^-|_{2_\alpha^*}^{2_\alpha^*}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
I(su^+ + tu^-) &+ \frac{as^2}{2}A_1 + \frac{bs^4}{2}A_1\|u^+\|^2 + \frac{bs^4}{4}A_1^2 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\\
&+ \frac{at^2}{2}A_2 + \frac{bt^4}{2}A_2\|u^-\|^2 + \frac{bt^4}{4}A_2^2 - \frac{t^{2_\alpha^*}}{2_\alpha^*}B_2 \leq c^\mu
\end{aligned}
\tag{2.17}
$$

for all $s \geq 0$ and all $t \geq 0$.

In the following, we shall proceed through three steps to complete the proof.

**Step 1 : we prove that $\mathbf{u}^{\pm} \neq \mathbf{0}$.**

Fist, we prove $u^+ \neq 0$. By contradiction, we suppose $u^+ = 0$. So, let $t = 0$ in (2.17), we have that

$$
\frac{as^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1 \leq c^\mu, \tag{2.18}
$$

for all $s \geq 0$.

**Case 1** $B_1 = 0$.

If $A_1 = 0$, that is, $u_n^+ \to u^+$ in $X_0$. According to Lemma 2.2, we obtain $\|u^+\| > 0$, which contradicts our supposition. If $A_1 > 0$, by (2.18), we have that

$$\frac{as^2}{2}A_1 + \frac{bs^4}{4}A_1^2 \le c^\mu$$

for all $s \ge 0$, which is absurd. Anyway, we have a contradiction.

**Case 2** $B_1 > 0$.

According to definition of $S$, we have that

$$\delta = \frac{\alpha}{N} S^{\frac{N}{2\alpha}} \le \frac{\alpha}{N}\Big(\frac{A_1}{(B_1)^{\frac{2}{2_\alpha^*}}}\Big)^{\frac{N}{2\alpha}}.$$

It is easy to see that

$$\frac{\alpha}{N}\left(\frac{A_1}{(B_1)^{\frac{2}{2_\alpha^*}}}\right)^{\frac{N}{2\alpha}} = \max_{s\ge 0}\left\{\frac{as^2}{2}A_1 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\right\} \le \max_{s\ge 0}\left\{\frac{as^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\right\}.$$

Thanks to $c^\mu < \delta$ and (2.18), we have that

$$\delta \le \max_{s\ge 0}\left\{\frac{as^2}{2}A_1 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\right\} \le \max_{s\ge 0}\left\{\frac{as^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\right\} < \delta.$$

That is, we obtain a contradiction.

From above discussions, we have that $u^+ \ne 0$.

Similarly, we can prove $u^- \ne 0$.

**Step 2 : we prove that $\mathbf{B_1 = B_2 = 0}$.**

Since the situation $B_2 = 0$ is analogous, we only prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$.

**Case 1** $B_2 > 0$.

Let $\widetilde{s}$ and $\widetilde{t}$ satisfy

$$\frac{a\widetilde{s}^2}{2}A_1 + \frac{b\widetilde{s}^4}{4}A_1^2 - \frac{\widetilde{s}^{2_\alpha^*}}{2_\alpha^*}B_1 = \max_{s\ge 0}\left\{\frac{as^2}{2}A_1 + \frac{bs^4}{4}A_1^2 - \frac{s^{2_\alpha^*}}{2_\alpha^*}B_1\right\},$$

$$\frac{a\widetilde{t}^2}{2}A_2 + \frac{b\widetilde{t}^4}{4}A_2^2 - \frac{\widetilde{t}^{2_\alpha^*}}{2_\alpha^*}B_2 = \max_{t\ge 0}\left\{\frac{at^2}{2}A_2 + \frac{bt^4}{4}A_2^2 - \frac{t^{2_\alpha^*}}{2_\alpha^*}B_2\right\}.$$

Because $[0,\widetilde{s}] \times [0,\widetilde{t}]$ is compact and $\psi_u$ is continuous, there are $(s_u, t_u) \in [0,\widetilde{s}] \times [0,\widetilde{t}]$ such that

$$\psi_u(s_u, t_u) = \max_{(s,t)\in[0,\widetilde{s}]\times[0,\widetilde{t}]} \psi_u(s,t).$$

In the following, we prove that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \widetilde{t})$.

If $t$ small enough, we have that

$$\psi_u(s, 0) = I(su^+) < I(su^+) + I(tu^-) \leq I(su^+ + tu^-) = \psi_u(s, t),$$

for all $s \in [0, \widetilde{s}]$.

So, there is $t_0 \in [0, \widetilde{t}]$ such that

$$\psi_u(s, 0) \leq \psi_u(s, t_0),$$

for all $s \in [0, \widetilde{s}]$.

That is, any point of $(s, 0)$ with $0 \leq s \leq \widetilde{s}$ is not the maximizer of $\psi_u$. Hence $(s_u, t_u) \notin [0, \widetilde{s}] \times \{0\}$. By similar discussions, we obtain $(s_u, t_u) \notin \{0\} \times [0, \widetilde{s}]$.

On the other hand, we have that

$$\frac{as^2}{2} A_1 + \frac{bs^4}{2} A_1 \|u^+\|^2 + \frac{bs^4}{4} A_1^2 - \frac{s^{2^*_\alpha}}{2^*_\alpha} B_1 > 0, \tag{2.19}$$

$$\frac{at^2}{2} A_2 + \frac{bt^4}{2} A_2 \|u^-\|^2 + \frac{bt^4}{4} A_2^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha} B_2 > 0, \tag{2.20}$$

$s \in (0, \widetilde{s}]$, $t \in (0, \widetilde{t}]$.

Hence,

$$\begin{aligned} \delta \leq & \frac{a\widetilde{s}^2}{2} A_1 + \frac{b\widetilde{s}^4}{4} A_1^2 - \frac{\widetilde{s}^{2^*_\alpha}}{2^*_\alpha} B_1 + \frac{b\widetilde{s}^4}{2} A_2 \|u^+\|^2 \\ & + \frac{at^2}{2} A_2 + \frac{bt^4}{2} A_2 \|u^-\|^2 + \frac{bt^4}{4} A_2^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha} B_2, \end{aligned}$$

$$\begin{aligned} \delta \leq & \frac{a\widetilde{t}^2}{2} A_2 + \frac{b\widetilde{t}^4}{4} A_2^2 - \frac{\widetilde{t}^{2^*_\alpha}}{2^*_\alpha} B_2 + \frac{b\widetilde{t}^4}{2} A_2 \|u^-\|^2 \\ & + \frac{as^2}{2} A_1 + \frac{bs^4}{2} A_1 \|u^+\|^2 + \frac{bs^4}{4} A_1^2 - \frac{s^{2^*_\alpha}}{2^*_\alpha} B_1 \end{aligned}$$

for all $s \in [0, \widetilde{s}]$ and all $t \in [0, \widetilde{t}]$.

Therefore, by (2.17), we have that

$$\psi_u(s, \widetilde{t}) \leq 0, \; \psi_u(\widetilde{s}, t) \leq 0$$

for all $s \in [0, \widetilde{s}]$ and all $t \in [0, \widetilde{t}]$.

So, $(s_u, t_u) \notin \{\widetilde{s}\} \times [0, \widetilde{t}]$ and $(s_u, t_u) \notin \times[0, \widetilde{s}] \times \{\widetilde{t}\}$.

From above discussions, we obtain $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \widetilde{t})$. Hence, we conclude that $(s_u, t_u)$ is a critical point of $\psi_u$.

That is, $s_u u^+ + t_u u^- \in \mathcal{M}$.

Therefore, by (2.17), (2.19) and (2.20), it follows from definition of $c^\mu$ that

$$\begin{aligned} c^\mu &\geq I(s_u u^+ + t_u u^-) + \frac{a s_u^2}{2} A_1 + \frac{b s_u^4}{2} A_1 \|u^+\|^2 + \frac{b s_u^4}{4} A_1^2 - \frac{s_u^{2_\alpha^*}}{2_\alpha^*} B_1 \\ &\quad + \frac{a t_u^2}{2} A_2 + \frac{b t_u^4}{2} A_2 \|u^-\|^2 + \frac{b t_u^4}{4} A_2^2 - \frac{t_u^{2_\alpha^*}}{2_\alpha^*} B_2 \\ &> I(s_u u^+ + t_u u^-) \geq c^\mu. \end{aligned}$$

That is, we have a contradiction.

**Case 2** $B_2 = 0$.

In this case, we can maximize in $[0, \widetilde{s}] \times [0, \infty)$. Indeed, it is possible to show that there exist $t_0 \in [0, \infty)$ such that $I(su^+ + tu^-) \leq 0$, for all $(s, t) \in [0, \widetilde{s}] \times [t_0, \infty)$. Hence, there are $(s_u, t_u) \in [0, \widetilde{s}] \times [0, \infty)$ satisfy

$$\psi_u(s_u, t_u) = \max_{s \in [0, \widetilde{s}] \times [0, \infty)} \psi_u(s, t).$$

Now, we prove that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \infty)$.

It is easy to see that $\psi_u(s, 0) < \psi_u(s, t)$ for $s \in [0, \widetilde{s}]$ and $t$ small enough, so we have $(s_u, t_u) \notin [0, \widetilde{s}] \times \{0\}$.

At the same time, $\psi_u(0, t) < \psi_u(s, t)$ for $t \in [0, \infty)$ and $s$ small enough, then we have $(s_u, t_u) \notin \{0\} \times [0, \infty)$.

On the other hand, we have that

$$\begin{aligned} \delta \leq \frac{a \widetilde{s}^2}{2} A_1 + \frac{b \widetilde{s}^4}{4} A_1^2 - \frac{\widetilde{s}^{2_\alpha^*}}{2_\alpha^*} B_1 + \frac{b \widetilde{s}^4}{2} A_2 \|u^+\|^2 \\ + \frac{a t^2}{2} A_2 + \frac{b t^4}{2} A_2 \|u^-\|^2 + \frac{b t^4}{4} A_2^2, \end{aligned}$$

for all $t \in [0, \infty)$.

Hence, we have that $\psi_u(\widetilde{s}, t) \leq 0$ for all $t \in [0, \infty)$. Thus, $(s_u, t_u) \notin \{\widetilde{s}\} \times [0, \infty)$. And so $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \infty)$. So, $(s_u, t_u)$ is an inner maximizer of $\psi_u$ in $[0, \widetilde{s}) \times [0, \infty)$. That is, $s_u u^+ + t_u u^- \in \mathcal{M}$.

Therefore, thanks to (2.19), it follows from definition of $c^\mu$ that

$$\begin{aligned} c^\mu &\geq I(s_u u^+ + t_u u^-) + \frac{a s_u^2}{2} A_1 + \frac{b s_u^4}{2} A_1 \|u^+\|^2 + \frac{b s_u^4}{4} A_1^2 - \frac{s_u^{2_\alpha^*}}{2_\alpha^*} B_1 \\ &\quad + \frac{a t_u^2}{2} A_2 + \frac{b t_u^4}{2} A_2 \|u^-\|^2 + \frac{b t_u^4}{4} A_2^2 \\ &> I(s_u u^+ + t_u u^-) \geq c^\mu, \end{aligned}$$

which is a contradiction.

Therefore, from above discussions, we obtain $B_1 = B_2 = 0$.

**Step 3 : we prove that c̄ is achieved**.

For $u^{\pm} \neq 0$, according to Lemma 2.2, there exist $s_u, t_u > 0$ such that $\bar{u} := s_u u^+ + t_u u^- \in \mathcal{M}$.

At the same time, by $B_1 = B_2 = 0$, it is easy to see that $\langle I'(u), u^{\pm}\rangle \leq 0$. So, from Lemma 2.2 again, we have that $0 < s_u, t_u \leq 1$.

Therefore, in view of $u_n \in \mathcal{M}$, (2.15), $B_1 = B_2 = 0$ and the norm in $X_0$ is lower semicontinuous, ones has

$$\begin{aligned}
c^{\mu} &\leq I(\bar{u})) - \frac{1}{4}\langle I'(\bar{u}), \bar{u}\rangle \\
&= \frac{a}{4}\|\bar{u}\|^2 + \left(\frac{1}{4} - \frac{1}{2^*_\alpha}\right)|\bar{u}|^{2^*_\alpha}_{2^*_\alpha} + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(\bar{u})\bar{u} - 4F(\bar{u})]dx \\
&= \frac{a}{4}(\|s_u u^+\|^2 + \|t_u u^-\|^2) + \left(\frac{1}{4} - \frac{1}{2^*_\alpha}\right)(|s_u u^+|^{2^*_\alpha}_{2^*_\alpha} + |t_u u^-|^{2^*_\alpha}_{2^*_\alpha}) \\
&\quad + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(s_u u^+)(s_u u^+) - 4F(s_u u^+)]dx + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(t_u u^-)(t_u u^-) \\
&\quad - 4F(t_u u^-)]dx \\
&\leq \frac{a}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{2^*_\alpha}\right)|u|^{2^*_\alpha}_{2^*_\alpha} + \frac{\mu}{4}\int_{\mathbb{R}^3}[f(u)u - 4F(u)]dx \\
&\leq \liminf_{n\to\infty}[I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle] \\
&= \liminf_{n\to\infty} I(u_n) \\
&= c^{\mu}.
\end{aligned}$$

So, $s_u = t_u = 1$ and $c^{\mu}$ is achieved by $u := u^+ + u^- \in \mathcal{M}$. □

## 3 The Proof of Main Results

Firstly, we prove Theorem 1.1.

***Proof of Theorem 1.1*** Since $u \in \mathcal{M}$, we have $\langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0$. By Lemma 2.2, for $(s, t) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1, 1)$, we have

$$I(su^+ + tu^-) < I(u^+ + u^-) = c^{\mu}. \tag{3.1}$$

If $I'(u) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|I'(v)\| \geq \theta, \text{ for all } \|v - u\| \leq 3\delta.$$

Choose $\sigma \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u\|}\})$. Let $D := (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $k(s,t) = su^+ + tu^-$, $(s,t) \in D$. It follows from (3.1) that

$$\overline{c}_\mu := \max_{\partial D} I_b \circ k < c^\mu. \tag{3.2}$$

Let $\varepsilon := min\{(c^\mu - \overline{c}_\mu)/2, \theta\delta/8\}$ and $S_\delta := B(u,\delta)$, in view of Lemma 2.3 in [54], there is a deformation $\eta \in C([0,1] \times X_0, X_0)$ satisfies

(a) $\eta(1,v) = v$ if $v \notin I^{-1}([c^\mu - 2\varepsilon, c^\mu + 2\varepsilon] \cap S_{2\delta})$;
(b) $\eta(1, I^{c^\mu+\varepsilon} \cap S_\delta) \subset I^{c^\mu-\varepsilon}$;
(c) $I(\eta(1,v)) \leq I(v)$ for all $v \in X_0$.

According to Lemma 2.2 and $(b)$, we can obtain

$$\max_{(s,t)\in\bar{D}} I(\eta(1, k(s,t))) < c^\mu. \tag{3.3}$$

Now, we prove that $\eta(1, k(D)) \cap \mathcal{M} \neq \varnothing$.
Let $\gamma(s,t) := \eta(1, k(s,t))$ and

$$\begin{aligned}\Psi_0(s,t) &:= (\langle I'(k(s,t)), u^+\rangle, \langle I'(k(s,t)), u^-\rangle)\\ &= (\langle I'(su^+ + tu^-), u^+\rangle, \langle I'(su^+ + tu^-), u^-\rangle)\\ &:= (g_1(s,t), g_1(s,t))\end{aligned}$$

and

$$\Psi_1(s,t) := \Big(\frac{1}{s}\langle I'(\gamma(s,t)), (\gamma(s,t))^+\rangle, \frac{1}{t}\langle I'(\gamma(s,t)), (\gamma(s,t))^-\rangle\Big).$$

It is noticed that, by $(f_3)$, we have

$$f'(s)s^2 - 3f(s)s > 0 \text{ for } s \neq 0. \tag{3.4}$$

Let

$$M = \begin{bmatrix} \frac{\partial g_1(s,t)}{\partial s}|_{(1,1)} & \frac{\partial g_2(s,t)}{\partial s}|_{(1,1)} \\ \frac{\partial g_1(s,t)}{\partial t}|_{(1,1)} & \frac{\partial g_2(s,t)}{\partial t}|_{(1,1)} \end{bmatrix},$$

then it follows from $u \in \mathcal{M}$ and (3.4) that

$$\det M = \frac{\partial g_1(s,t)}{\partial s}|_{(1,1)} \times \frac{\partial g_2(s,t)}{\partial t}|_{(1,1)} - \frac{\partial g_1(s,t)}{\partial t}|_{(1,1)} \times \frac{\partial g_2(s,t)}{\partial s}|_{(1,1)} > 0.$$

Therefore, by degree theory, we conclude that $\Psi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, so that $\eta(1, k(s_0, t_0)) = \gamma(s_0, t_0) \in \mathcal{M}$. That is, we get a contradiction with (3.3).
Therefore, $u$ is a sign-changing solution for problem (1.1). □

Next, we prove that the energy of $u$ is strictly larger than that of the ground state energy.

***Proof of Theorem 1.2*** With a similar argument to the proof of Lemma 2.4, we can obtain that there is $\mu_1^\star > 0$ such that for all $\mu \geq \mu_1^\star$, there exists $v \in \mathcal{N}$ such that $I(v) = c^* > 0$. By standard arguments, the critical points of the functional $I$ on $\mathcal{N}$ are critical points of $I$ in $X_0$ and we obtain $I'(v) = 0$. That is, $v$ is a ground state solution of problem (1.1).

From Theorem 1.1, for all $\mu \geq \mu^\star$, the problem (1.1) has a ground state sign-changing solution $u$.

Set $\mu^{\star\star} = \max\{\mu^\star, \mu_1^\star\}$. Similar as the proof of Lemma 2.2, there are $s_{u^+}, t_{u^-} \in (0, 1)$ such that

$$s_{u^+}u^+ \in \mathcal{N}, t_{u^-}u^- \in \mathcal{N}.$$

Therefore, in view of Lemma 2.2, we have that

$$2c^* \leq I(s_{u^+}u^+) + I(t_{u^-}u^-) \leq I(s_{u^+}u^+ + t_{u^-}u^-) < I(u^+ + u^-) = c^\mu.$$

Meanwhile, we conclude that $c^*$ cannot be achieved by a sign-changing function. □

***Remark 3.1*** We believe that, by using the abstract tools contained in [15,16], some additional existence results for Kirchhoff-type equation can be obtained. Furthermore, in our opinion, the same strategy can be useful studying a wide class of elliptic problems.

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