International Journal of Bifurcation and Chaos, Vol. 31, No. 2 (2021) 2150019 (24 pages)
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DOI: 10.1142/S021812742150019X

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Dynamics of a Food Chain Model with Two Infected Predators

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Received November 17, 2019; Revised April 25, 2020

In this paper, the dynamics of a three-species food chain model with two predators infected by an infectious disease is investigated. The positivity and boundedness of the system, the existence of the equilibria and the basic reproductive number are given. Sufficient conditions for the local stability of all equilibria are obtained by analyzing the corresponding characteristic equations. By constructing suitable Lyapunov functions and taking the geometric approach, the global stability of all equilibria is proved. According to the center manifold theory, this model undergoes the phenomenon of backward and forward bifurcations in a certain range of the basic reproductive number R_0. By taking the disease transmission coefficient of predator as bifurcation parameter, Hopf bifurcation emerges in the neighborhood of the endemic equilibrium. Furthermore, the optimal control of the disease is discussed by the Pontryagin's maximum principle. Various simulations are given to support the analytical results.

Keywords: Food chain; backward and forward bifurcation; Hopf bifurcation; global stability; Pontryagin's maximum principle.

1. Introduction

In recent years, the food chain models including three populations are widely investigated (see [Holmes & Bethel, 1972; Gard & Hallam, 1979; Fryxell & Lundberg, 1997; Hsu *et al.*, 2015; Jiang *et al.*, 2017; Dutta *et al.*, 2017; Meng *et al.*, 2011; Ma *et al.*, 2020; Ma & Zhang, 2018; Meng & Wu, 2018; Zeng *et al.*, 2018; Yu *et al.*, 2019; Meng & Wu, 2020]). For example, Meng and Wu [2018] investigated a phytoplankton–zooplankton-fish model with nonlinear fish harvesting and taxation. However, diseases are not to be ignored in ecological population. Most of the ecological populations suffer from various infectious diseases which have a significant influence on population size. Infectious diseases may disrupt the homeostatic state of an ecosystem, that is, the stability of the system may be destroyed and the species may become extinct as time goes on. Thus, some scholars have expanded their interests in diseases in different ecological systems, which is known as eco-epidemiological. Anderson and May [1986] first investigated an eco-epidemiological model where the predator interacts with infected prey species. Then, this problem attracted much more attention from many experts (see [Haladar *et al.*, 2015; Xu & Zhang, 2013; Das, 2016; Kant & Kumar, 2017; Wang & Feng, 2015; Hao *et al.*, 2016; Mbava *et al.*, 2017; Meng & Li, 2020] and references therein). The dynamics of an infectious disease transmission modified

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Leslie–Gower type eco-epidemiological model in both deterministic and stochastic fluctuating environment was considered by Haladar *et al.* [2015]. Global stability of a predator–prey system with a transmissible disease in the predator population and a time delay representing the gestation period of the predator was studied by Xu and Zhang [2013]. Das [2016] described a predator–prey system with disease in both populations.

The functional response is a response in which the predation rate of each predator per unit time varies with the density of its prey, that is, the predation effect of predators on prey. Depending upon type of species and their living environment, the predation terms are of different forms where the amount of food consumed by the predator is not only related to the prey density but also to the capture ability of the predator. Recently, many scholars have preferred to study the predator–prey systems with functional response (see [Meng *et al.*, 2011; Wang & Wei, 2015; Zhang & Sun, 2005; Tewa *et al.*, 2013; Pei *et al.*, 2005; Meng *et al.*, 2018; Meng & Wu, 2018; Meng & Wang, 2019; Jia *et al.*, 2019; Huo *et al.*, 2019; Yang & Wang, 2020]). For instance, Meng *et al.* [2011] investigated a three-species system with Holling-II type functional response and feedback delays. Wang and Wei [2015] proposed a predator–prey system with strong Allee effect and an Ivlve-type functional response. Zhang and Sun [2005] studied the permanence of a predator–prey system with disease in the predator and Holling-II type functional response.

Some authors considered that the predators only catch the infected prey. Johri *et al.* [2012] proposed a Lotka–Volterra type predator–prey model with disease in prey. Meng *et al.* [2018] investigated a predator–prey system with harvesting prey and disease in prey species. Sharma and Samanta [2015] presented an eco-epidemiological system with two preys, one predator and disease in the first prey species. However, the predators cannot distinguish between the infected and susceptible prey, so the predators eat both the infected prey and the susceptible prey. Rossi *et al.* [2015] investigated a three-level food chain where a transmissible disease spreads only among the bottom prey species and the predator consumes the infected and susceptible bottom prey. In addition, most of the scholars considered that the predators are not infected through infected prey in eco-epidemiological system. But, it is not always true from the biological viewpoint. For instance, a predator–prey system where both population are subjected to parasitism has been proposed and analyzed by Hadeler and Freedman [1989]. They assumed that the predators become infected by feeding on the infected prey.

In our model, because the susceptible super-predator can distinguish the infected predator and susceptible predator, the susceptible super-predator captures not only the susceptible predator but also the infective predator based on the work [Rossi *et al.*, 2015]. In addition, due to the super-predator becoming infected by feeding on the infected prey [Hadeler & Freedman, 1989], we will divide the super-predator population into two compartments: susceptible super-predator and infective super-predator, where infective super-predator has no capture ability. Based on the above analysis, we will propose a three-species food chain model with Holling-II type functional response and disease in the predator and super-predator. Detailed assumptions are listed in the next section.

The rest of the paper is organized as follows: In Sec. 2, we will formulate an eco-epidemiological model and discuss the basic properties of such model. In Sec. 3, by analyzing the corresponding characteristic equations, we will prove the local asymptotic stability of the system around equilibria. By constructing the suitable Lyapunov functions and the geometric approach, we will also prove the global stability of all equilibria. In Sec. 4, we will study backward bifurcation, forward bifurcation and Hopf bifurcation. By using the classical method of Pontryagin's maximum principle, the optimal control strategies will be discussed in Sec. 5. To support our theoretical analysis, some numerical simulations will be given in Sec. 6. We end with a brief conclusion and discussion in the last section.

2. A Three-Species Food Chain Eco-Epidemiological Model

2.1. *Model description*

We consider a three-species food chain eco-epidemiological model. We assume that $X(t)$, $P(t)$ and $Q(t)$ are the density of prey, predator and super-predator at time t, respectively. The ecological system is based on the following assumptions.

(A_1) In the absence of predator and super-predator, the prey population $X(t)$ grows logistically with the intrinsic growth rate r and carrying capacity r/a.

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The prey is consumed by the predator with Holling-I functional response.

(A_2) Predator population is divided into two classes: susceptible predator $P_1(t)$ and infective predator $P_2(t)$. Therefore, the total of the predator population is $P_1(t) + P_2(t)$. The disease spreads among the predator population only by contact, but cannot be transmitted vertically. The infected predators do not recover or become immune. The disease incidence is assumed to be the simple mass action incidence $\beta P_1 P_2$, where $\beta > 0$ is called the coefficient of disease transmission. The susceptible predator is captured by the super-predator with the linear function. The infective predator is captured by the super-predator with the Holling-II function.

(A_3) Only the susceptible predator and super-predator have capture ability, and the susceptible super-predators have the ability to distinguish the susceptible predator and infective predator because the susceptible predators are much stronger than the infected predators. Therefore, the super-predator can catch not only the susceptible predator but also the infected predator. The super-predator becomes the infected super-predator by feeding on the infected predator. Thus the super-predator population is also composed of two parts: susceptible super-predator $Q_1(t)$ and infective super-predator $Q_2(t)$. Then, the total biomass of the super-predator is $Q_1(t) + Q_2(t)$.

(A_4) The infected predator and infected super-predator populations have their mortality due to infection.

From the aforementioned assumptions, we can get the model structure shown in Fig. 1 and the corresponding system

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = X(r - aX) - a_1 X P_1, \\ \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = e_1 a_1 X P_1 - a_2 P_1 Q_1 - \beta P_1 P_2 - d_1 P_1, \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = \beta P_1 P_2 - \dfrac{a_3 P_2 Q_1}{1 + bP_2} - (d_2 + c_1) P_2, \\ \dfrac{\mathrm{d}Q_1}{\mathrm{d}t} = e_2 a_2 P_1 Q_1 + \dfrac{e_3 a_3 P_2 Q_1}{1 + bP_2} \\ \qquad\qquad - \dfrac{m a_3 P_2 Q_1}{1 + bP_2} - d_3 Q_1, \\ \dfrac{\mathrm{d}Q_2}{\mathrm{d}t} = \dfrac{m a_3 P_2 Q_1}{1 + bP_2} - (d_4 + c_2) Q_2, \end{cases} \tag{1}$$

with the initial conditions

$$X(0) > 0, \quad P_1(0) > 0, \quad P_2(0) > 0,$$
$$Q_1(0) > 0, \quad Q_2(0) > 0.$$

We assume that all parameters of system (1) are positive. The biological meanings of parameters are given in Table 1.

Notice that the first, second, third and fourth equations of system (1) are independent of the variable $Q_2(t)$. Therefore, we can consider the following system:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = X(r - aX) - a_1 X P_1, \\ \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = e_1 a_1 X P_1 - a_2 P_1 Q_1 - \beta P_1 P_2 - d_1 P_1, \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = \beta P_1 P_2 - \dfrac{a_3 P_2 Q_1}{1 + bP_2} - (d_2 + c_1) P_2, \\ \dfrac{\mathrm{d}Q_1}{\mathrm{d}t} = e_2 a_2 P_1 Q_1 + \dfrac{e_3 a_3 P_2 Q_1}{1 + bP_2} \\ \qquad\qquad - \dfrac{m a_3 P_2 Q_1}{1 + bP_2} - d_3 Q_1, \end{cases} \tag{2}$$

and the initial conditions are

$$X(0) > 0, \quad P_1(0) > 0, \quad P_2(0) > 0, \quad Q_1(0) > 0.$$

The purpose of this paper is to study the dynamics of system (2).

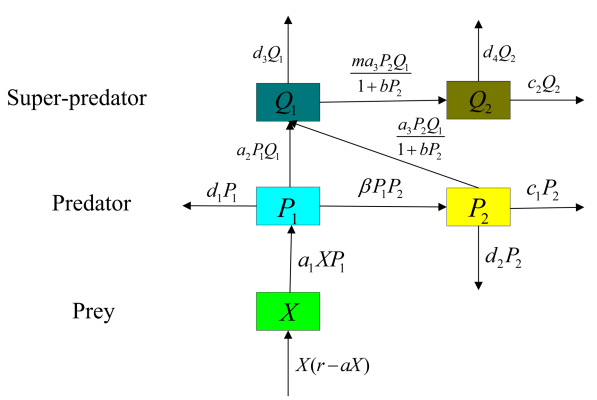

Fig. 1. Transfer diagram of eco-epidemiological model.

Table 1. Biological meanings of parameters.

Parameters	Biological Meanings
r	Intrinsic growth rate of the prey
a	The intra-specific competition rate of the prey population
a_1	Capture rate of the prey by the susceptible predator
a_2	Capture rate of the susceptible predator by the susceptible super-predator
a_3	Capture rate of the infected predator by the susceptible super-predator
β	The disease transmission coefficient
e_1	Conversion coefficient from prey to susceptible predator
e_2	Conversion coefficient from susceptible predator to susceptible super-predator
e_3	Conversion coefficient from infected predator to susceptible super-predator
d_1	Natural death rate of the susceptible predator
d_2	Natural death rate of the infected predator
d_3	Natural death rate of the susceptible super-predator
d_4	Natural death rate of the infected super-predator
c_1	Death rate of the infected predator due to infection
c_2	Death rate of the infected super-predator due to infection
m	Conversion coefficient from susceptible super-predator to infected super-predator
b	The half-saturation constant

2.2. *Basic properties*

In this subsection, we will investigate the positivity and boundedness of the solutions of system (2) with the initial conditions.

Lemma 1. *Every solution of system (2) with the initial conditions exists in the interval* $[0, \infty)$, *and* $X(t) > 0, P_1(t) > 0, P_2(t) > 0, Q_1(t) > 0$, *for all* $t \geq 0$.

Proof. Since the right-hand side of system (2) is completely continuous and locally Lipschitz on $C(\mathbb{R}^4_+)$, here $\mathbb{R}^4_+ = \{(X(t), P_1(t), P_2(t), Q_1(t)) : X(t) \geq 0, P_1(t) \geq 0, P_2(t) \geq 0, Q_1(t) \geq 0\}$, the solution $(X(t), P_1(t), P_2(t), Q_1(t))$ of system (2) with the initial conditions exists and is unique on $[0, \zeta]$, here $0 < \zeta \leq +\infty$ [Hale, 1971]. From the initial conditions of system (2), we have

$$X(t) = X(0) \exp\left[\int_0^t \{r - aX(\theta) - a_1 P_1(\theta)\} \mathrm{d}\theta\right] > 0,$$

$$P_1(t) = P_1(0) \exp\left[\int_0^t \{e_1 a_1 X(\theta) - a_2 Q_1(\theta) - \beta P_2(\theta) - d_1\} \mathrm{d}\theta\right] > 0,$$

$$P_2(t) = P_2(0) \exp\left[\int_0^t \left\{\beta P_1(\theta) - \frac{a_3 Q_1(\theta)}{1 + bP_2(\theta)} - (d_2 + c_1)\right\} \mathrm{d}\theta\right] > 0,$$

$$Q_1(t) = Q_1(0) \exp\left[\int_0^t \left\{e_2 a_2 P_1(\theta) + \frac{e_3 a_3 P_2(\theta)}{1 + bP_2(\theta)} - \frac{m a_3 P_2(\theta)}{1 + bP_2(\theta)} - d_3\right\} \mathrm{d}\theta\right] > 0,$$

which completes the proof. ■

Lemma 2. *Positive solutions of system (2) with the initial conditions are ultimately bounded.*

Proof. Let $(X(t), P_1(t), P_2(t), Q_1(t))$ be any positive solution of system (2) with the initial conditions. Define a function

$$W(t) = X(t) + P_1(t) + P_2(t) + Q_1(t).$$

Calculating the derivative of $W(t)$ along positive solutions of system (2), it follows that

$$\begin{aligned}
\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \frac{\mathrm{d}X(t)}{\mathrm{d}t} + \frac{\mathrm{d}P_1(t)}{\mathrm{d}t} + \frac{\mathrm{d}P_2(t)}{\mathrm{d}t} + \frac{\mathrm{d}Q_1(t)}{\mathrm{d}t} \\
&= rX(t) - aX^2(t) + a_1 X P_1(t)(e_1 - 1) \\
&\quad + \frac{a_3 P_2(t) Q_1(t)}{1 + bP_2(t)}(e_3 - m - 1) \\
&\quad + a_2 P_1(t) Q_1(t)(e_2 - 1) - d_1 P_1(t) \\
&\quad - (d_2 + c_1) P_2(t) - d_3 Q_1(t).
\end{aligned}$$

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Now we choose μ such that $\mu = \min\{d_1, d_2 + c_1, d_3\}$ and $0 < e_i < 1$ $(i = 1, 2, 3)$. Then the above inequality reduces to

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} &\le rX(t) - aX^2(t) - \mu(W(t) - X(t)) \\ &= -a\left(X(t) - \frac{r+\mu}{2a}\right)^2 - \mu W(t) + \frac{(r+\mu)^2}{4a} \\ &\le \frac{(r+\mu)^2}{4a} - \mu W(t),\end{aligned}$$

which yields

$$\limsup_{t\to\infty} W(t) \le \frac{(r+\mu)^2}{4a\mu}.$$

We observe that all the solutions of system (2) initiating in $\mathbb{R}^4_+$ eventually lie in the region Ω defined by

$$\Omega = \left\{(X(t), P_1(t), P_2(t), Q_1(t)) \in \mathbb{R}^4_+ : W(t) \le \frac{(r+\mu)^2}{4a\mu}\right\},$$

which is a positively invariant set for system (2). So we will study system (2) in the set Ω. ■

3. Stability Analysis of All Equilibria

3.1. *Equilibria and the basic reproductive number*

In this subsection, we will investigate the existence of the equilibria and the basic reproductive number of system (2). Firstly, we calculate the basic reproductive number R_0 of system (2) by using the next-generation method [van den Driessche & Watmough, 2002]. Here, we have the following matrix of new infection $\mathcal{F}(x)$, and the matrix of transfer $\mathcal{V}(x)$. Let $x = (X, P_1, P_2, Q_1)^T$. Thus, system (2) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$\mathcal{F}(x) = \begin{pmatrix} \beta P_1 P_2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}(x) = \begin{pmatrix} \dfrac{a_3 P_2 Q_1}{1 + bP_2} + (d_2 + c_1)P_2 \\ -e_2 a_2 P_1 Q_1 + \dfrac{(m - e_3)a_3 P_2 Q_1}{1 + bP_2} + d_3 Q_1 \\ -rX + aX^2 + a_1 X P_1 \\ -e_1 a_1 X P_1 + a_2 P_1 Q_1 + \beta P_1 P_2 + d_1 P_1 \end{pmatrix}.$$

The Jacobian matrices of $\mathcal{F}(x)$ and $\mathcal{V}(x)$ at the disease-free equilibrium E_4 are, respectively,

$$D\mathcal{F}(E_4) = \begin{pmatrix} \dfrac{\beta d_3}{e_2 a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D\mathcal{V}(E_4) = \begin{pmatrix} v_{11} & 0 & 0 & 0 \\ v_{21} & v_{22} & 0 & -e_2 a_2 Q_1 \\ 0 & 0 & v_{33} & a_1 X \\ \beta P_1 & a_2 P_1 & -e_1 a_1 P_1 & v_{44} \end{pmatrix},$$

here $v_{11} = a_3 Q_1 + (d_2 + c_1)$, $v_{21} = -e_3 a_3 Q_1 + m a_3 Q_1$, $v_{22} = -e_2 a_2 P_1 + d_3$, $v_{33} = -r + 2aX + a_1 P_1$, $v_{44} = -e_1 a_1 X + a_2 Q_1 + d_1$.

The reproduction number, denoted by R_0 is given by

$$R_0 = \frac{\beta a a_2 d_3}{e_2 a_2 a_3 (r e_1 a_1 - a d_1) - e_1 a_1^2 a_3 d_3 + a e_2 a_2^2 (d_2 + c_1)}.$$

In order for the convenience of computation, some notations are also given as follows

$$R_{00} = \frac{r e_1 a_1}{a d_1}, \quad R_{01} = \frac{\beta r}{a_1 (d_2 + c_1)},$$

$$R_{03} = \frac{r e_2 a_2}{a_1 d_3}, \quad R_{02} = \frac{\beta r e_1 a_1}{\beta a d_1 + \beta e_1 a_1^2 (d_2 + c_1)},$$

$$R_{04} = \frac{r e_1 e_2 a_1 a_2}{e_1 a_1^2 d_3 + a e_2 a_2 d_1}.$$

Next, we will study the existence of the boundary equilibria of system (2). Biologically, system (2) may have the following boundary equilibria.

The trivial equilibrium is $E_0(0, 0, 0, 0)$. The prey-only equilibrium is $E_1(\frac{r}{a}, 0, 0, 0)$. The equilibrium is $E_2(X^{(2)}, P_1^{(2)}, 0, 0)$, where $X^{(2)} = \frac{d_1}{e_1 a_1}$, $P_1^{(2)} = \frac{r e_1 a_1 - a d_1}{e_1 a_1^2}$, and the existence condition of this equilibrium is $R_{00} > 1$.

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The super-predator free equilibrium is $E_3(X^{(3)}, P_1^{(3)}, P_2^{(3)}, 0)$, where

$$X^{(3)} = \frac{\beta r - a_1(d_2 + c_1)}{\beta a}, \quad P_1^{(3)} = \frac{d_2 + c_1}{\beta},$$

$$P_2^{(3)} = \frac{\beta r e_1 a_1 - \beta a d_1 - \beta e_1 a_1^2 (d_2 + c_1)}{\beta^2 a}.$$

This equilibrium exists if $R_{01} > 1$ and $R_{02} > 1$ hold.

The disease-free equilibrium is $E_4(X^{(4)}, P_1^{(4)}, 0, Q_1^{(4)})$, where

$$X^{(4)} = \frac{r e_2 a_2 - a_1 d_3}{a e_2 a_2}, \quad P_1^{(4)} = \frac{d_3}{e_2 a_2},$$

$$Q_1^{(4)} = \frac{r e_1 e_2 a_1 a_2 - e_1 a_1^2 d_3 - a e_2 a_2 d_1}{a e_2 a_2^2},$$

and the existence conditions of the disease-free equilibrium are $R_{03} > 1$ and $R_{04} > 1$.

Lastly, we will calculate the endemic equilibrium of system (2). The endemic equilibrium $E^*(X^*, P_1^*, P_2^*, Q_1^*)$ of system (2) can be deduced by the following equations

$$\begin{cases} X^*(r - aX^*) - a_1 X^* P_1^* = 0, \\ e_1 a_1 X^* P_1^* - a_2 P_1^* Q_1^* - \beta P_1^* P_2^* - d_1 P_1^* = 0, \\ \beta P_1^* P_2^* - \dfrac{a_3 P_2^* Q_1^*}{1 + bP_2^*} - (d_2 + c_1) P_2^* = 0, \\ e_2 a_2 P_1^* Q_1^* + \dfrac{e_3 a_3 P_2^* Q_1^*}{1 + bP_2^*} - \dfrac{m a_3 P_2^* Q_1^*}{1 + bP_2^*} - d_3 Q_1^* = 0. \end{cases} \tag{3}$$

From the fourth equation of (3), we have

$$P_1^* = \frac{d_3(1 + bP_2^*) - a_3 P_2^*(e_3 - m)}{e_2 a_2 (1 + bP_2^*)}.$$

Substituting P_1^* into the first equation of (3), we can obtain

$$X^* = \frac{(1 + bP_2^*)(r e_2 a_2 - a_1 d_3) + a_1 a_3 (e_3 - m) P_2^*}{a a_2 e_2 (1 + bP_2^*)}.$$

Substituting P_1^* into the third equation of (3), we can obtain

$$Q_1^* = \frac{(1 + bP_2^*)[\beta d_3 - e_2 a_2 (d_2 + c_1)] - \beta a_3 (e_3 - m) P_2^*}{e_2 a_2 a_3}.$$

Substituting X^*, P_1^* and Q_1^* into the second equation of (3), we have a quadratic equation of P_2^* as follows

$$\overline{m}_1 (P_2^*)^2 + \overline{m}_2 P_2^* + \overline{m}_3 = 0,$$

where

$$\begin{aligned} \overline{m}_1 &= e_2 a a_2^2 b^2 (d_2 + c_1) + \beta a a_2 a_3 b (e_3 - e_2 - m) \\ &\quad - \beta a a_2 b^2 d_3, \\ \overline{m}_2 &= 1 - R_0, \\ \overline{m}_2 &= b(1 - R_0) + a_3 (e_3 - m)(e_1 a_1^2 a_3 + \beta a a_2) \\ &\quad - \beta a a_2 (b d_2 + e_2 a_3) + e_2 a a_2^2 b (d_2 + c_1). \end{aligned}$$

It is easy to show that if the following assumption

$$\begin{aligned} \text{(H1)} \quad & r e_2 a_2 (1 + bP_2^*) + a_1 a_3 (e_3 - m) P_2^* \\ & > a_1 d_3 (1 + bP_2^*), \\ & d_3 (1 + bP_2^*) > a_3 P_2^* (e_3 - m), \\ & \beta d_3 (1 + bP_2^*) > a_2 e_2 (d_2 + c_1)(1 + bP_2^*) \\ & \qquad + \beta a_3 (e_3 - m) P_2^* \end{aligned}$$

holds, then $X^* > 0$, $P_1^* > 0$ and $Q_1^* > 0$. Thus, the following results are established.

Theorem 1. *The system (2) has*

(i) *a unique endemic equilibrium* $E_1^*(X^*, P_1^*, P_2^*, Q_1^*)$ *when* $R_0 > 1$;
(ii) *a unique endemic equilibrium* E_2^* *when* $m_2 < 0$ *and* $m_3 = 0$ *or* $\triangle = \overline{m}_2^2 - 4\overline{m}_1\overline{m}_3 = 0$;
(iii) *two different endemic equilibria* E_3^* *and* E_4^* *when* $m_2 < 0$, $R_0 < 1$ *and* $\triangle = \overline{m}_2^2 - 4\overline{m}_1\overline{m}_3 > 0$;
(iv) *no endemic equilibrium under the otherwise cases.*

Remark 3.1. In what follows, we will only discuss the unique endemic equilibrium E_1^*, which is written as $E^*(X^*, P_1^*, P_2^*, Q_1^*)$. Similar method can be used for the other cases.

3.2. *Local stability of all equilibria*

In this part, we will investigate the local stability of system (2) in the neighborhood of the equilibria by analyzing the corresponding characteristic equations. The Jacobian matrix of system (2) is given

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by

$$J=\begin{pmatrix} a_{11} & -a_1X & 0 & 0 \\ e_1a_1P_1 & a_{22} & -\beta P_1 & -a_2P_1 \\ 0 & \beta P_2 & a_{33} & -\dfrac{a_3P_2}{1+bP_2} \\ 0 & ea_2Q_1 & \dfrac{(e_3-m)a_3Q_1}{(1+bP_2)^2} & a_{44} \end{pmatrix},$$

where

$$a_{11}=r-2aX-a_1P_1,$$
$$a_{22}=e_1a_1X-a_2Q_1-\beta P_2-d_1,$$
$$a_{33}=\beta P_1-\frac{a_3Q_1}{(1+bP_2)^2}-(d_2+c_1),$$
$$a_{44}=e_2a_2P_1-d_3+\frac{(e_3-m)a_3P_2}{1+bP_2}.$$

With help of this matrix, we will analyze the local stability of equilibria of system (2).

It is easy to calculate that eigenvalues of the Jacobian matrix at E_0 are

$$\lambda_1=r,\quad \lambda_2=-d_1,$$
$$\lambda_3=-(d_2+c_1),\quad \lambda_4=-d_3.$$

Hence, the trivial equilibrium E_0 is not a stable equilibrium. In fact, it is a saddle point.

The eigenvalues of the Jacobian matrix at E_1 are

$$\lambda_1=-r,\quad \lambda_2=\frac{re_1a_1-ad_1}{a},$$
$$\lambda_3=-(d_2+c_1),\quad \lambda_4=-d_3.$$

It is clear that the prey-only equilibrium E_1 is locally asymptotically stable when $R_{00}<1$.

The eigenvalues of the Jacobian matrix at E_2 are $\lambda_1=\beta P_1^{(2)}-(d_2+c_1)$ and $\lambda_2=e_3a_3P_1^{(2)}-d_3$, and the others satisfying the following equation

$$\lambda^2+\tilde{a}_1\lambda+\tilde{a}_2=0,$$

here $\tilde{a}_1=a_1X^{(2)}$ and $\tilde{a}_2=e_1a_1^2X^{(2)}P_1^{(2)}$. It is clear that the equilibrium E_2 is locally asymptotically stable when $R_{02}<1$ and $R_{04}<1$.

One eigenvalue of the Jacobian matrix at E_3 is $\lambda_1=e_2a_2P_1^{(3)}+\frac{e_3a_3P_2^{(3)}}{1+bP_2^{(3)}}-\frac{ma_3P_2^{(3)}}{1+bP_2^{(3)}}-d_3$, and the other eigenvalues satisfy the equation

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0,$$

where $b_1=aX^{(3)}$, $b_2=e_1a_1^2X^{(3)}P_1^{(3)}+\beta^2P_1^{(3)}P_2^{(3)}$ and $b_3=\beta^2aX^{(3)}P_1^{(3)}P_2^{(3)}$. It is clear that $b_1>0$, $b_2>0$ and $b_3>0$. According to the Routh–Hurwitz criterion, the super-predator-free equilibrium E_3 is locally asymptotically stable when $b_1b_2>b_3$ and $e_2a_2P_1^{(3)}+\frac{e_3a_3P_2^{(3)}}{1+bP_2^{(3)}}<\frac{ma_3P_2^{(3)}}{1+bP_2^{(3)}}+d_3$.

From the above discussion, we have the following results.

Theorem 2

(i) *The trivial equilibrium E_0 is not locally asymptotically stable for all parameters, and the prey-only equilibrium E_1 is locally asymptotically stable when $R_{00}<1$.*

(ii) *The boundary equilibrium E_2 is locally asymptotically stable when $R_{02}<1$ and $R_{04}<1$, and the super-predator-free equilibrium E_3 is locally asymptotically stable when $b_1b_2>b_3$ and $e_2a_2P_1^{(3)}+\frac{e_3a_3P_2^{(3)}}{1+bP_2^{(3)}}<\frac{ma_3P_2^{(3)}}{1+bP_2^{(3)}}+d_3$.*

In addition, we have the following results on the local stability of the disease-free equilibrium E_4 and the endemic equilibrium E_1^*.

Theorem 3

(i) *If $R_{03}>1$, $R_{04}>1$, $R_0<1$ and $\tilde{d}_1\tilde{d}_2>\tilde{d}_3$, then the disease-free equilibrium E_4 is locally asymptotically stable.*

(ii) *If $R_0>1$ and the assumption*

$$\text{(H2)}\quad m_1m_2-m_3>0\quad\text{and}$$
$$m_3(m_1m_2-m_3)-m_1^2m_4>0$$

hold on, then the endemic equilibrium E_1^* is locally asymptotically stable.

Proof

(i) One of the eigenvalues of the Jacobian matrix at E_4 is $\lambda_1=\beta P_1^{(4)}-a_3Q_1^{(4)}-(d_2+c_1)$, and the others satisfy the following equation

$$\lambda^3+\tilde{d}_1\lambda^2+\tilde{d}_2\lambda+\tilde{d}_3=0, \tag{4}$$

here $\tilde{d}_1=aX^{(4)}>0$, $\tilde{d}_2=e_1a_1^2X^{(4)}P_1^{(4)}+e_2a_2^2P_1^{(4)}Q_1^{(4)}>0$ and $\tilde{d}_3=ae_2a_2^2X^{(4)}P_1^{(4)}Q_1^{(4)}>0$ when $R_{03}>1$ and $R_{04}>1$. If $\tilde{d}_1\tilde{d}_2>\tilde{d}_3$, then all eigenvalues of the Eq. (4) are negative by the Routh–Hurwitz criterion. That is, the disease-free equilibrium E_4 is locally asymptotically stable if $R_{03}>1$, $R_{04}>1$, $R_0<1$ and $\tilde{d}_1\tilde{d}_2>\tilde{d}_3$.

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(ii) The characteristic equation of the linearization of system (2) at the endemic equilibrium E_1^* is

$$\begin{vmatrix} \lambda + aX^* & a_1X^* & 0 & 0 \\ -e_1a_1P_1^* & \lambda & \beta P_1^* & a_2P_1^* \\ 0 & -\beta P_2^* & \lambda & \dfrac{a_3P_2^*}{1+bP_2^*} \\ 0 & -e_2a_2Q_1^* & \dfrac{(m-e_3)a_3Q_1^*}{(1+bP_2^*)^2} & \lambda \end{vmatrix} = 0.$$

Simplifying the above determinant, the characteristic equation can be rewritten as

$$F(\lambda) = \lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4 = 0, \quad (5)$$

where

$$\begin{aligned} m_1 &= aX^*, \\ m_2 &= e_2a_2^2P_1^*Q_1^* + \beta^2P_1^*P_2^* + \frac{(e_3-m)a_3^2}{(1+bP_2^*)^2}P_2^*Q_1^* \\ &\quad + e_1a_1^2X^*P_1^*, \\ m_3 &= \left(\frac{\beta a_2a_3(e_3-m)}{(1+bP_2^*)^2} - \frac{\beta e_2a_2a_3}{1+bP_2^*}\right)P_1^*P_2^*Q_1^* \\ &\quad + ae_2a_2^2X^*P_1^*Q_1^* + \beta^2aX^*P_1^*P_2^* \\ &\quad + \frac{a(e_3-m)a_3^2}{(1+bP_2^*)^3}X^*P_2^*Q_1^*, \\ m_4 &= \left(-\frac{\beta ae_2a_2a_3}{1+bP_2^*} + \frac{\beta aa_2a_3(e_3-m)}{(1+bP_2^*)^2}\right. \\ &\quad \left. + \frac{e_1a_1^2a_2a_3(e_3-m)}{(1+bP_2^*)^3}\right)X^*P_1^*P_2^*Q_1^*. \end{aligned}$$

It is assumed that $m_i > 0$ $(i = 1, 2, 3, 4)$. According to the Routh–Hurwitz criterion, E_1^* is locally asymptotically stable if the assumption (H2) holds on. ■

Remark 3.2. Despite the complexity of system (2), we can find that the endemic equilibrium E_2^* is a saddle-node point, E_3^* is an unstable node point and E_4^* is a stable node point. Some related proofs are omitted here in theory.

3.3. *Global stability of all equilibria*

In this part, we will not only study the local asymptotic stability of endemic equilibrium E_1^* of system (2), but also investigate the global asymptotic stability of the three subsystems of system (2) around the boundary equilibria E_2, E_3 and E_4, respectively.

Theorem 4. *The subsystem of system (2) with*

(i) *the prey and the susceptible predator populations are globally asymptotically stable around the equilibrium* $E_2(X^{(2)}, P_1^{(2)}, 0, 0)$;
(ii) *the prey, the susceptible predator and the infected predator population are globally asymptotically stable around the equilibrium* $E_3(X^{(3)}, P_1^{(3)}, P_2^{(3)}, 0)$;
(iii) *the prey, the susceptible predator and the susceptible super-predator population are globally asymptotically stable around the equilibrium* $E_4(X^{(4)}, P_1^{(4)}, 0, Q_1^{(4)})$.

Proof

(i) Let us consider a suitable Lyapunov function

$$V_1(X, P_1) = e_1\int_{X^{(2)}}^{X}\frac{x-X^{(2)}}{x}\mathrm{d}x + \int_{P_1^{(2)}}^{P_1}\frac{y-P_1^{(2)}}{y}\mathrm{d}y.$$

It is easy to verify that the function $V_1(X^{(2)}, P_1^{(2)}) = 0$ and $V_1(X, P_1) > 0$ for all the other positive values $X^{(2)}$ and $P_1^{(2)}$.

Differentiating V_1 along the solutions of the respective subsystem of system (2) with respect to t and using the fact $P_2 = 0$, $Q_1 = 0$, we can obtain

$$\begin{aligned} \frac{\mathrm{d}V_1}{\mathrm{d}t} &= e_1\frac{X-X^{(2)}}{X}\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{P_1-P_1^{(2)}}{P_1}\frac{\mathrm{d}P_1}{\mathrm{d}t} \\ &= e_1(X-X^{(2)})[-a(X-X^{(2)}) - a_1(P_1-P_1^{(2)})] \\ &\quad + e_1a_1(X-X^{(2)})(P_1-P_1^{(2)}) \\ &= -e_1a(X-X^{(2)})^2 \le 0. \end{aligned}$$

When $X = X^{(2)}$, $P_1 = P_1^{(2)}$, $\frac{dV_1}{dt} = 0$. Therefore, the largest invariant set at which $\frac{dV_1}{dt} = 0$ is the equilibrium $E_2(X^{(2)}, P_1^{(2)}, 0, 0)$. By LaSalle's invariance principle [LaSalle, 1968], subsystem of system (2) with the prey and the susceptible predator population is globally asymptotically stable at the equilibrium $E_2(X^{(2)}, P_1^{(2)}, 0, 0)$.

Similarly, we can prove the other results. ■

In order to prove the global stability of the endemic equilibrium of system (2), we need the

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following results, which can be found in reference [Thieme, 1992].

We will get simple sufficient conditions that the endemic equilibrium E_1^* is globally asymptotically stable when $R_0 > 1$ by using the geometrical approach of Li and Muldowney [1995]. Above all, we give a brief outline of this geometrical approach. Let $|x| \to f(x) \in R^n$ be a C^1 function for x in an open set $D \in R^n$. We consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{6}$$

and denote that $x(t, x_0)$ is the solution to (6) such that $x(0, x_0) = x_0$. We make the following two assumptions.

(H3) There exists a compact absorbing set $K \subset D$.
(H4) Equation (6) has a unique equilibrium $\overline{x}$ in D.

The equilibrium $\overline{x}$ is said to be globally stable in D, if it is locally stable and all trajectories in D converge to $\overline{x}$. For $n \geq 2$, by Bendixson's criterion we obtain a condition satisfied by f which precludes the existence of nonconstant periodic solutions of (6). The classical Bendixson's condition div $f(x) < 0$ for $n = 2$ is robust under C^1 local perturbations of f. For higher-dimensional systems, the robust properties of C^1 are discussed in reference [Hirsch, 1990].

Theorem 5. *If $R_0 > 1$, then the unique endemic equilibrium E_1^* of system (2) is globally asymptotically stable.*

Proof. For system (2), we will show the global stability of the endemic equilibrium E_1^* when $R_0 > 1$. Since $X + P_1 + P_2 + Q_1 \to \frac{(r+\mu)^2}{4a\mu}$ as $t \to \infty$, system (2) is a three-dimensional asymptotically autonomous differential system with the limit system

$$\begin{cases} \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = e_1 a_1 P_1 \left(\dfrac{(r+\mu)^2}{4a\mu} - P_1 - P_2 - Q_1 \right) \\ \qquad\qquad - a_2 P_1 Q_1 - \beta P_1 P_2 - d_1 P_1, \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = \beta P_1 P_2 - \dfrac{a_3 P_2 Q_1}{1 + bP_2} - (d_2 + c_1) P_2, \\ \dfrac{\mathrm{d}Q_1}{\mathrm{d}t} = e_2 a_2 P_1 Q_1 + \dfrac{e_3 a_3 P_2 Q_1}{1 + bP_2} \\ \qquad\qquad - \dfrac{m a_3 P_2 Q_1}{1 + bP_2} - d_3 Q_1. \end{cases} \tag{7}$$

The Jacobian matrix of system (7) is given by

$$J = \begin{pmatrix} b_{11} & -e_1 a_1 P_1 - \beta P_1 & -e_1 a_1 P_1 - a_2 P_1 \\ \beta P_2 & b_{22} & -\dfrac{a_3 P_2}{1 + bP_2} \\ e_2 a_2 Q_1 & \dfrac{a_3 Q_1 (e_3 - m)}{(1 + bP_2)^2} & b_{33} \end{pmatrix}, \tag{8}$$

where

$$b_{11} = \frac{e_1 a_1 (r+\mu)^2}{4a\mu} - e_1 a_1 (2P_1 + P_2 + Q_1) - a_2 Q_1 - \beta P_2 - d_1,$$

$$b_{22} = \beta P_1 - \frac{a_3 Q_1}{(1 + bP_2)^2} - (d_2 + c_1),$$

$$b_{33} = e_2 a_2 P_1 + \frac{a_3 P_2 (e_3 - m)}{1 + bP_2} - d_3.$$

Then the global stability analysis of E_1^* will be performed through the geometric approach due to [Li & Muldowney, 1995]. From the results of [Hirsch, 1990], we can see that the method requires the following sufficient conditions of the global stability of E_1^*:

(i) the uniqueness of E_1^* in the interior of Ω [i.e. condition (H3)],
(ii) the existence of an absorbing compact set in the interior of Ω [i.e. condition (H4)],
(iii) the fulfillment of a Bendixson's criterion (i.e. the inequality $\overline{q}_2 < 0$).

Under the assumption $R_0 > 1$, system (2) satisfies conditions (H3) and (H4). In fact, when $R_0 > 1$, the disease-free equilibrium E_4 is unstable. The instability of E_4 together with $E_4 \in \partial\Omega$ implies the uniform persistence [Freedman *et al.*, 1994]. That is, there exists a constant $c > 0$ such that

$$\lim_{t\to\infty} \inf X(t) > c, \quad \lim_{t\to\infty} \inf P_1(t) > c,$$

$$\lim_{t\to\infty} \inf P_2(t) > c, \quad \lim_{t\to\infty} \inf Q_1(t) > c.$$

The uniform persistence, because of boundedness of Ω, is equivalent to the existence of a compact set in the interior of Ω which is absorbing for (2) (see [Hutson & Schmitt, 1992]). Thus, (H3) is verified. Moreover, as previously shown, E_1^* is the only equilibrium in the interior of Ω, so that (H4) is also verified.

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Now, we need to find the conditions of the Bendixson's criterion. Taking into account the Jacobian matrix (8), we obtain the second additive compound matrix $J^{[2]}(X, P_1, P_2, Q_1)$

$$J^{[2]} = \begin{pmatrix} c_{11} & -\dfrac{a_3 P_2}{1+bP_2} & e_1 a_1 P_1 + a_2 P_1 \\ \dfrac{a_3 Q_1 (e_3 - m)}{(1+bP_2)^2} & c_{22} & -e_1 a_1 P_1 - \beta P_1 \\ -e_2 a_2 Q_1 & \beta P_2 & c_{33} \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} c_{11} &= \frac{e_1 a_1 (r+\mu)^2}{4a\mu} + e_1 a_1 (2P_1 + P_2) - \frac{a_3 Q_1}{(1+bP_2)^2} \\ &\quad + e_1 a_1 Q_1 - a_2 Q_1 + \beta(P_1 - P_2) \\ &\quad - (d_1 + d_2 + c_1), \\ c_{22} &= \frac{e_1 a_1 (r+\mu)^2}{4a\mu} + \frac{a_3 P_2 (e_3 - m)}{1+bP_2} \\ &\quad + e_1 a_1 (2P_1 + P_2 + Q_1) + e_2 a_2 P_1 \\ &\quad - a_2 Q_1 - \beta P_2 - (d_1 + d_3), \\ c_{33} &= \beta P_1 - \frac{a_3 Q_1}{(1+bP_2)^2} + \frac{a_3 P_2 (e_3 - m)}{1+bP_2} \\ &\quad + e_2 a_2 P_1 - (d_2 + d_3 + c_1). \end{aligned}$$

Set the function $p(x) = E(P_1, P_2, Q_1) = \mathrm{diag}\{\frac{P_2}{Q_1}, \frac{P_2}{Q_1}, \frac{P_2}{Q_1}\}$. Then $E_f E^{-1} = \mathrm{diag}\{\frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1}, \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1}, \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1}\}$. Incorporating (8), the matrix $B = E_f E^{-1} + E J^{[2]} E^{-1}$ can be written in block form as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$\begin{aligned} B_{11} &= \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} + \frac{e_1 a_1 (r+\mu)^2}{4a\mu} \\ &\quad - e_1 a_1 (2P_1 + P_2 + Q_1) - \frac{a_3 Q_1}{(1+bP_2)^2} \\ &\quad - a_2 Q_1 + \beta(P_1 - P_2) - (d_1 + d_2 + c_1), \\ B_{12} &= \left(-\frac{a_3 P_2}{1+bP_2}, e_1 a_1 P_1 + a_2 P_1 \right), \\ B_{21} &= \left(\frac{a_3 Q_1 (e_3 - m)}{(1+bP_2)^2}, -e_2 a_2 Q_1 \right)^T, \\ B_{22} &= \begin{pmatrix} d_{11} & -e_1 a_1 P_1 - \beta P_1 \\ \beta P_2 & d_{22} \end{pmatrix}, \end{aligned}$$

here

$$\begin{aligned} d_{11} &= \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} + \frac{e_1 a_1 (r+\mu)^2}{4a\mu} \\ &\quad - e_1 a_1 (2P_1 + P_2 + Q_1) + \frac{a_3 P_2 (e_3 - m)}{1+bP_2} \\ &\quad - a_2 Q_1 - \beta P_2 + e_2 a_2 P_2 - (d_1 + d_3), \\ d_{22} &= \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} - \beta P_1 - \frac{a_3 Q_1}{(1+bP_2)^2} + e_2 a_2 P_1 \\ &\quad + \frac{a_3 P_2 (e_3 - m)}{1+bP_2} - (d_2 + d_3 + c_1). \end{aligned}$$

Let $(\overline{u}, \overline{v}, \overline{w})$ be the vectors in $R^3 \simeq R\binom{n}{2}$. We select a norm in R^3 as $|(\overline{u}, \overline{v}, \overline{w})| = \max\{|\overline{u}|, |\overline{v}| + |\overline{w}|\}$. Furthermore, let $\mu(\cdot)$ be the Lozinskiĭ measure with respect to this norm. Following the method in [Martin, 1974] we can get

$$\mu(B) \leq \sup\{g_1, g_2\}, \tag{10}$$

where $g_1 = \mu(B_{11}) + |B_{12}|$, $g_2 = \mu(B_{22}) + |B_{21}|$. $|B_{12}|$ and $|B_{21}|$ are the matrix norms. More specifically,

$$\begin{aligned} \mu(B_{11}) &= \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} - e_1 a_1 (2P_1 + P_2 + Q_1) \\ &\quad - \frac{a_3 Q_1}{(1+bP_2)^2} + \frac{e_1 a_1 (r+\mu)^2}{4a\mu} \\ &\quad - a_2 Q_1 - \beta(P_2 - P_1) - (d_1 + d_2 + c_1), \\ |B_{12}| &= \max\left\{ -\frac{a_3 P_2}{1+bP_2}, e_1 a_1 P_1 + a_2 P_1 \right\} \\ &= e_1 a_1 P_1 + a_2 P_1, \\ |B_{21}| &= \max\left\{ \frac{a_3 Q_1 (e_3 - m)}{(1+bP_2)^2}, -e_2 a_2 Q_1 \right\} \\ &= \frac{a_3 Q_1 (e_3 - m)}{(1+bP_2)^2}. \end{aligned}$$

To calculate $\mu(B_{22})$, we add the absolute value of the off-diagonal elements to the diagonal one in each column of B_{22}, and then take the maximum of

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two sums. This leads to

$$\begin{aligned}\mu(B_{22}) = \max\Bigg\{&\frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} - e_1a_1(2P_1 + P_2 + Q_1) - a_2Q_1 + e_2a_2P_2 - (d_1 + d_3) + \frac{a_3P_2(e_3 - m)}{1 + bP_2} + \frac{e_1a_1(r + \mu)^2}{4a\mu},\\ &\frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} - e_1a_1P_1 - \frac{a_3Q_1}{(1 + bP_2)^2} + \frac{a_3P_2(e_3 - m)}{1 + bP_2} + e_2a_2P_1 - (d_2 + d_3 + c_1)\Bigg\}\\ = &\frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} + \frac{a_3P_2(e_3 - m)}{1 + bP_2} - d_3 + e_2a_2P_1 - e_1a_1P_1\\ &+ \max\Bigg\{-e_1a_1(P_1 + P_2 + Q_1) - a_2Q_1 + \frac{e_1a_1(r + \mu)^2}{4a\mu} - d_1, -\frac{a_3Q_1}{(1 + bP_2)^2} - (d_2 + c_1)\Bigg\}\\ = &\frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} + \frac{a_3P_2(e_3 - m)}{1 + bP_2} - d_3 + e_2a_2P_1 + \frac{e_1a_1(r + \mu)^2}{4a\mu} - 2e_1a_1P_1 - e_1a_1(P_2 + Q_1) - a_2Q_1 - d_1.\end{aligned}$$

From the second and the third equations of system (7), we have

$$\frac{\dot{P}_2}{P_2} = \beta P_1 - \frac{a_3Q_1}{1 + bP_2} - (d_2 + c_1), \quad \frac{\dot{Q}_1}{Q_1} = e_2a_2P_1 + \frac{e_3a_3P_2}{1 + bP_2} - \frac{ma_3P_2}{1 + bP_2} - d_3.$$

Therefore, we can get

$$\begin{aligned}g_1 &= \mu_1(B_{11}) + |B_{12}|\\ &= \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} + \frac{e_1a_1(r + \mu)^2}{4a\mu} - e_1a_1(2P_1 + P_2 + Q_1) - \frac{a_3Q_1}{(1 + bP_2)^2}\\ &\quad - a_2Q_1 - \beta(P_2 - P_1) + e_1a_1P_1 + a_2P_1 - (d_1 + d_2 + c_1)\\ &\le \frac{\dot{P}_2}{P_2} + \frac{e_1a_1(r + \mu)^2}{4a\mu} - a_2Q_1 - \frac{a_3Q_1}{(1 + bP_2)^2} + a_2P_1 - \beta(P_1 - P_2) - (d_1 + d_2 + c_1),\\ g_2 &= \mu(B_{22}) + |B_{21}|\\ &= \frac{\dot{P}_2}{P_2} - \frac{\dot{Q}_1}{Q_1} + \frac{a_2P_2(e_3 - m)}{1 + bP_2} + \frac{e_1a_1(r - \mu)^2}{4a\mu} - e_1a_1P_1 + e_2a_2P_1 - e_1a_1(P_1 + P_2 + Q_1)\\ &\quad - a_2Q_1 + \frac{a_3Q_1(e_3 - m)}{(1 + bP_2)^2} - (d_1 + d_3)\\ &\le \frac{\dot{P}_2}{P_2} + \frac{e_1a_1(r + \mu)^2}{4a\mu} - e_1a_1P_1 - a_1Q_1 + \frac{a_3Q_1(e_3 - m)}{(1 + bP_2)^2} - d_1.\end{aligned}$$

Hence by (10) we have

$$\begin{aligned}\mu(B) \le &\frac{\dot{P}_2}{P_2} + \frac{e_1a_1(r + \mu)^2}{4a\mu} - d_1\\ &+ \max\Bigg\{a_2P_1 - a_2Q_1 - \beta(P_1 - P_2) - \frac{a_3Q_1}{(1 + bP_2)^2} - (d_2 + c_1), \frac{a_3Q_1(e_3 - m)}{(1 + bP_2)^2} - e_1a_1P_1 - a_1Q_1\Bigg\}.\end{aligned}$$

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In view of $P_1 \leq \frac{(r+\mu)^2}{4a\mu}$, $P_2 \geq c$, $Q_1 \geq c$, here c is the constant of uniform persistence, we can deduce that if

$$\frac{e_1 a_1 (r+\mu)^2}{4a\mu} < \beta\left(\frac{(r+\mu)^2}{4a\mu} - c\right) + \frac{a_3 c}{(1+bc)^2} + d_1 + d_2 + c_1,$$

$$\frac{a_3 c(e_3 - m)}{(1+bc)^2} < a_1 c + d_1,$$

holds, then

$$\mu(B) \leq \frac{\dot{P}_2}{P_2} - d,$$

where

$$d = \min\left\{\beta\left(\frac{(r+\mu)^2}{4a\mu} - c\right) + \frac{a_3 c}{(1+bc)^2} + d_1 + d_2 + c_1 - \frac{e_1 a_1 (r+\mu)^2}{4a\mu}, a_1 c + d_1 - \frac{a_3 c(e_3 - m)}{(1+bc)^2}\right\} > 0.$$

Along each solution $(P_1(t), P_2(t), Q_1(t))$ of system (7) for which $(P_1(0), P_2(0), Q_1(0)) \in \Omega$, we have

$$\begin{aligned}\frac{1}{t}\int_0^t \mu(B)ds &\leq \frac{1}{t}\int_0^{t_1} \mu(B)ds + \frac{1}{t}\int_{t_1}^{t_2} \mu(B)ds \\ &\leq \frac{1}{t}\int_0^{t_1} \mu(B)ds + \frac{1}{t}\log\frac{P_2(t)}{P_2(t_1)} - d,\end{aligned}$$

which implies that

$$\bar{q}_2 = \lim_{t\to\infty} \sup_{x_0 \in \Omega} \frac{1}{t}\int_0^t \mu(B(x(s, x_0)))ds \leq -\frac{d}{2} < 0.$$

According to the results in [Hirsch, 1990] and Theorem 3(ii), if $R_0 > 1$, then the endemic equilibrium E_1^* of system (2) is globally stable in Ω. ■

4. Analysis of Bifurcation

4.1. *Backward bifurcation and forward bifurcation*

Theorem 3 shows that $R_0 = 1$ is a bifurcation value. In fact, the disease-free equilibrium changes its stability properties when the basic reproduction number R_0 passes through the value 1. In order to look for conditions on the parameter values causing a forward bifurcation or a backward bifurcation, we will make use of the result in [Castillo-Chavez & Song, 2004], which is based on the general center manifold theory [Guckenheimer & Holmes, 1983].

We consider a general system of ODEs with a parameter ϕ:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \phi), \tag{11}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n$, is continuously differentiable at least twice in both x and ϕ. Without loss of generality, we assume that $x = 0$ is an equilibrium of system (11) for all values of the parameter ϕ. That is, $f(0, \phi) \equiv 0$, for all ϕ. In what follows, we give one important result [Castillo-Chavez & Song, 2004].

Lemma 3 [Castillo-Chavez & Song, 2004]. *Assume that*

(B1) $A = D_x f(0,0)$ *is the linearization matrix of system (11) around the equilibrium* $x = 0$ *with* ϕ *evaluated at* 0. *Zero is a simple eigenvalue of* A *and all other eigenvalues have negative real parts.*

(B2) *Matrix* A *has a (non-negative) right eigenvector* $\mathbf{w}$ *and a left eigenvector* $\mathbf{v}$ *corresponding to the zero eigenvalue.*

Let f_k *denote the* k*th component of* f, *and*

$$\bar{a} = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0),$$

$$\bar{b} = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0).$$

Then the local dynamics of system (11) around $x = 0$ *are totally determined by* $\bar{a}$ *and* $\bar{b}$.

(1) $\bar{a} > 0$, $\bar{b} > 0$. *When* $\phi < 0$ *and* $|\phi| \ll 1$, $x = 0$ *is locally asymptotically stable and there exists a positive unstable equilibrium; when* $0 < \phi \ll 1$, $x = 0$ *is unstable and there exists a negative and locally asymptotically stable equilibrium.*

(2) $\bar{a} < 0$, $\bar{b} < 0$. *When* $\phi < 0$ *and* $|\phi| \ll 1$, $x = 0$ *is unstable; when* $0 < \phi \ll 1$, $x = 0$ *is locally asymptotically stable and there exists a positive unstable equilibrium.*

(3) $\bar{a} > 0$, $\bar{b} < 0$. *When* $\phi < 0$ *and* $|\phi| \ll 1$, $x = 0$ *is unstable and there exists a locally asymptotically stable negative equilibrium; when* $0 < \phi \ll 1$, $x = 0$ *is stable and a positive unstable equilibrium appears.*

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(4) $\overline{a} < 0$, $\overline{b} > 0$. *When ϕ changes from negative to positive, $x = 0$ changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes positive and locally asymptotically stable.*

It clearly appears that a transcritical bifurcation takes place when $\phi = 0$. More precisely, when $\overline{a} < 0$ and $\overline{b} > 0$, such bifurcation is forward; when $\overline{a} > 0$ and $\overline{b} > 0$, this bifurcation at $\phi = 0$ is backward.

Now let $\phi = \beta$ be the bifurcation parameter, such that $R_0 < 1$ for $\phi < 0$ and $R_0 > 1$ for $\phi > 0$, and such that x_0 is a disease-free equilibrium for all values of ϕ. The disease-free equilibrium is the point $(x_0; \phi)$ and the local stability of the disease-free equilibrium changes at the point $(x_0; \phi)$ [van den Driessche & Watmough, 2002]. Now we want to show that there are nontrivial equilibria near the bifurcation point $(x_0; 0)$. Let $X = x_1$, $P_1 = x_2$, $P_2 = x_3$, and $Q_1 = x_4$, then system (2) can be written as

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1(r - ax_1) - a_1x_1x_2, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = e_1a_1x_1x_2 - a_2x_2x_4 - \beta x_2x_3 - d_1x_2, \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = \beta x_2x_3 - \dfrac{a_3x_3x_4}{1 + bx_3} - (d_2 + c_1)x_3, \\ \dfrac{\mathrm{d}x_4}{\mathrm{d}t} = e_2a_2x_2x_4 + \dfrac{e_3a_3x_3x_4}{1 + bx_3} - \dfrac{ma_3x_3x_4}{1 + bx_3} - d_3x_4. \end{cases} \tag{12}$$

We will apply Lemma 3 to show that system (12) may exhibit a backward bifurcation when $R_0 = 1$. We consider that the disease-free equilibrium E_4 is $E_4 = (x_1^{(4)}, x_2^{(4)}, x_3^{(4)}, x_4^{(4)})$. We observe that the condition $R_0 = 1$ can be given in terms of the parameter β while

$$\beta = \beta^* = \frac{re_1e_2a_1a_2a_3 - e_1a_1^2a_3d_3 - ae_2a_2a_3d_1 + ae_2a_2^2(d_2 + c_1)}{aa_2d_3}.$$

Therefore, when $\beta = \beta^*$, the characteristic matrix of the linearization of system (12) at the disease-free equilibrium E_4 is

$$J(E_4, \beta^*) = \begin{pmatrix} -ax_1^{(4)} & a_1x_1^{(4)} & 0 & 0 \\ e_1a_1x_2^{(4)} & 0 & -\beta^*x_2^{(4)} & -a_2x_2^{(4)} \\ 0 & 0 & 0 & 0 \\ 0 & e_2a_2x_4^{(4)} & e_3a_3x_4^{(4)} - ma_3x_4^{(4)} & 0 \end{pmatrix}.$$

Through simplification, the characteristic equation corresponding to the Jacobian matrix $J(E_4, \beta^*)$ at E_4 can be reduced to

$$\lambda(\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3) = 0, \tag{13}$$

where $C_1 = ax_1^{(4)}$, $C_2 = e_1a_1^2x_1^{(4)}x_2^{(4)} + e_2a_2^2x_1^{(4)}x_4^{(4)}$ and $C_3 = ae_2a_2^2x_1^{(4)}x_2^{(4)}x_4^{(4)}$.

Thus, a simple zero eigenvalue of (13) exists and the other eigenvalues are real and negative when $C_1C_2 > C_3$. By using the center manifold theory, the disease-free equilibrium E_4 is a non-hyperbolic equilibrium when $\beta = \beta^*$. Hence, the assumption (B1) of Lemma 3 is also verified.

Now we denote

$$\mathbf{w} = (w_1, w_2, w_3, w_4)^T$$

as a right eigenvector associated with the zero eigenvalue. It follows that

$$\begin{pmatrix} -ax_1^{(4)} & a_1x_1^{(4)} & 0 & 0 \\ e_1a_1x_2^{(4)} & 0 & -\beta^*x_2^{(4)} & -a_2x_2^{(4)} \\ 0 & 0 & 0 & 0 \\ 0 & e_2a_2x_4^{(4)} & e_3a_3x_4^{(4)} - ma_3x_4^{(4)} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = 0.$$

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Then we can get

$$\mathbf{w} = \left(\frac{(e_3 - m)a_1 a_3}{a e_2 e_3}, \frac{(m - e_3)a_3}{e_2 a_2}, 1, \frac{e_1 a_1^2 a_3 (e_3 - m)}{a e_2 a_2^2} - \frac{\beta^*}{a_2} \right)^T.$$

Furthermore, the components of the left eigenvector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ must satisfy the equalities

$$\begin{cases} -a x_1^{(4)} v_1 + e_1 a_1 x_2^{(4)} v_2 = 0, \\ -a_1 x_1^{(4)} v_1 + e_2 a_2 x_4^{(4)} v_4 = 0, \\ -\beta^* x_2^{(4)} v_2 + (e_3 a_3 x_4^{(4)} - m a_3 x_4^{(4)}) v_4 = 0, \\ -a_2 x_2^{(4)} v_2 = 0 \end{cases} \tag{14}$$

and $\mathbf{w} \cdot \mathbf{v} = 1$. By simple computation, we can get $\mathbf{v} = (0, 0, 1, 0)$.

According to the coefficients $\bar{a}$ and $\bar{b}$ defined in Lemma 3 and substituting the values of all the second-order derivatives evaluated at the disease-free equilibrium and $\beta = \beta^*$, we have

$$\begin{aligned} \bar{a} &= \sum_{k,i,j=1}^{4} v_k \omega_i \omega_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_4, \beta^*) \\ &= v_3 \bigg(w_2 w_3 \frac{\partial^2 f_3}{\partial x_2 \partial x_3} + w_3 w_2 \frac{\partial^2 f_3}{\partial x_3 \partial x_2} \\ &\quad + w_3 w_4 \frac{\partial^2 f_3}{\partial x_3 \partial x_4} + w_4 w_3 \frac{\partial^2 f_3}{\partial x_4 \partial x_3} + w_3^2 \frac{\partial^2 f_3}{\partial x_2^2} \bigg) \\ &\quad + 2 w_3 (\beta^* w_2 - a_3 w_4 + a_3 x_4 b w_3) \\ &= \frac{2 a_3}{a e_2 a_2^2} (\beta^* a a_2 (m - e_3) - e_1 a_1^2 a_3 (e_3 - m) \\ &\quad + \beta^* a e_2 a_2 + b (r e_1 e_2 a_1 a_2 - e_1 a_1^2 d_3 - a e_2 a_2 d_1)) \\ &= \frac{2 a_3 d_3}{a e_2 a_2^2} [-(e_3 - m)(r e_1 e_2 a_1 a_2 a_3 - e_1 a_1^2 a_3 d_3 \\ &\quad - a e_3 a_2 a_3 d_1 + a e_2 a_2^2 (d_2 + c_1) + e_1 a_1^2 a_3 d_3) \\ &\quad + e_2 (r e_1 e_2 a_1 a_2 a_3 - e_1 a_1^2 a_3 d_3 - a e_2 a_2 a_3 d_1 \\ &\quad + a e_2 a_2^2 (d_2 + c_1)) + b d_3 (r e_1 e_2 a_1 a_2 \\ &\quad - e_1 a_1^2 d_3 - a e_2 a_3 d_1)] \end{aligned}$$

and

$$\begin{aligned} \bar{b} &= \sum_{k,i=1}^{n} v_k \omega_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(E_4, \beta^*) \\ &= v_3 \bigg(w_2 \frac{\partial^2 f_3}{\partial x_2 \partial \beta} + w_3 \frac{\partial^2 f_3}{\partial x_3 \partial \beta} \bigg) \\ &= v_3 (w_2 x_3 + w_3 x_2) \\ &= \frac{d_3}{e_2 a_2} > 0. \end{aligned}$$

Note that the coefficient $\bar{b}$ is always positive. According to Lemma 3, the sign of the coefficient $\bar{a}$ decides the local dynamics around the disease-free equilibrium for $\beta = \beta^*$.

Define

$$R_1^* = \frac{R_{10}^*}{R_{100}^*},$$

where

$$\begin{aligned} R_{10}^* &= (e_3 - m)(e_1 a_1^2 a_3 d_3 + a e_2 a_2 a_3 d_1) \\ &\quad + r e_1 e_2^2 a_1 a_2 a_3 + a e_2^2 (d_2 + c_1) \\ &\quad + r e_1 e_2 a_1 a_2 b d_3 - e_1 a_1^2 b d_3^2, \\ R_{100}^* &= (e_3 - m)(r e_1 e_2 a_1 a_2 a_3 + a e_2 a_2^2 (d_2 + c_1) \\ &\quad + e_1 a_1^2 a_3 d_3) + e_2 a_3 (e_1 a_1^2 d_3 \\ &\quad + a e_2 a_2 d_1 + a b d_1 d_3). \end{aligned}$$

Furthermore, when $R_1^* < 1$, then $\bar{a} < 0$; when $R_1^* > 1$, then $\bar{a} > 0$. From the above discussions, we have the following result.

Theorem 6. *If $R_1^* > 1$, then system (2) exhibits a backward bifurcation when $R_0 = 1$. If $R_1^* < 1$, then system (2) exhibits a forward bifurcation when $R_0 = 1$.*

4.2. *Hopf bifurcation*

In this subsection, we choose transmission coefficient from the susceptible intermediate predator compartment to the infective intermediate predator compartment as the bifurcation parameter, and study the existence of Hopf bifurcation.

Theorem 7. *Hopf bifurcation occurs at the endemic equilibrium E_1^* when $\beta = \beta^{**}$ if and only if*

(i) *there exists a critical value $\beta = \beta^{**}$ such that $f(\beta^{**}) = m_1(\beta^{**}) m_2(\beta^{**}) m_3(\beta^{**}) - m_3^2(\beta^{**}) - m_4(\beta^{**}) m_1^2(\beta^{**}) = 0$;*
(ii) *$\rho(\beta)$ is purely imaginary number at $\beta = \beta^{**}$, but all other eigenvalues are real negative parts;*
(iii) *$L(\beta^{**}) N(\beta^{**}) + K(\beta^{**}) M(\beta^{**}) \neq 0$, here $L(\beta), K(\beta), M(\beta)$ and $N(\beta)$ are defined in (20).*

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Proof. By the condition $f(\beta^{**})$, the characteristic equation can be written as

$$\left(\lambda^2 + \frac{m_3}{m_1}\right)\left(\lambda^2 + m_1\lambda + \frac{m_1 m_4}{m_3}\right) = 0. \quad (15)$$

If Eq. (15) has four roots, defined as λ_i $(i = 1, 2, 3, 4)$ with the pair of purely imaginary roots at $\beta = \beta^{**}$ as $\lambda_1 = \overline{\lambda}_2$, then it follows that

$$\lambda_3 + \lambda_4 = m_1, \quad (16)$$

$$\omega_0^2 + \lambda_3\lambda_4 = m_2 = \frac{m_1^2 m_4 + m_3^2}{m_1 m_3}, \quad (17)$$

$$\omega_0^2(\lambda_3 + \lambda_4) = -m_3, \quad (18)$$

$$\omega_0^2\lambda_3\lambda_4 = m_4, \quad (19)$$

here $\omega_0 = \operatorname{Im}\lambda_1(\beta^{**})$. By the aforementioned equation (15), $\omega_0 = \sqrt{\frac{m_3}{m_1}}$. Now, if λ_3 and λ_4 are complex conjugate, then it follows that $2\operatorname{Re}\lambda_3 = -m_1$ from (16). If they are real roots, then $\lambda_3 < 0$ and $\lambda_4 < 0$ according to Eqs. (16) and (17). To complete the discussion, we need to verify the transversality condition.

Because $f(\beta^{**})$ is a continuous function of all its roots, there exists an open interval $\beta \in (\beta^{**} - \epsilon, \beta^{**} + \epsilon)$, where $\lambda_1(\beta)$ and $\lambda_2(\beta)$ are complex conjugates on β. We suppose that their general forms in this neighborhood are

$$\lambda_1(\beta) = a(\beta) + ib(\beta), \quad \lambda_2(\beta) = a(\beta) - ib(\beta).$$

Now, we verify the transversality condition

$$\left.\frac{\mathrm{d}\operatorname{Re}(\lambda_j(\beta))}{\mathrm{d}\beta}\right|_{\beta=\beta^{**}} \neq 0, \quad j = 1, 2.$$

Substituting $\lambda_j(\beta) = a(\beta) \pm ib(\beta)$ into (5) and calculating the derivative, we have

$$K(\beta)a'(\beta) - L(\beta)b'(\beta) + M(\beta) = 0,$$
$$L(\beta)a'(\beta) + M(\beta)b'(\beta) + N(\beta) = 0,$$

where

$$\begin{aligned} K(\beta) &= 4a^3(\beta) - 12a(\beta)b^2(\beta) \\ &\quad + 3m_1(\beta)(a^2(\beta) - b^2(\beta)) \\ &\quad + 2m_2(\beta)a(\beta) + m_3(\beta), \\ L(\beta) &= 12a^2(\beta)b(\beta) + 6m_1(\beta)a(\beta)b(\beta) \\ &\quad - 4b^3(\beta) + 2m_2(\beta)b(\beta), \\ M(\beta) &= m_1'(\beta)(a^3(\beta) - 3ab^2(\beta)) \\ &\quad + m_2'(\beta)(a^2(\beta) - b^2(\beta)) + m_3'(\beta)a(\beta), \\ N(\beta) &= m_1'(\beta)(3a^2(\beta)b(\beta) - b^3(\beta)) \\ &\quad + 2m_2'(\beta)a(\beta)b(\beta) + m_3'(\beta)b(\beta). \end{aligned} \quad (20)$$

Since

$$L(\beta^{**})N(\beta^{**}) + K(\beta^{**})M(\beta^{**}) \neq 0,$$

we have

$$\begin{aligned} &\left.\frac{\mathrm{d}\operatorname{Re}(\lambda_j(\beta))}{\mathrm{d}\beta}\right|_{\beta=\beta^{**}} \\ &\quad = -\frac{L(\beta^{**})N(\beta^{**}) + K(\beta^{**})M(\beta^{**})}{K^2(\beta^{**}) + L^2(\beta^{**})} \\ &\quad \neq 0, \quad j = 1, 2. \end{aligned}$$

Therefore, the transversality condition is true. This implies that Hopf bifurcation occurs at $\beta = \beta^{**}$. ■

5. Optimal Control Problem

5.1. *The existence of optimal control*

In the previous section, we investigate the dynamical behavior of this system. However, we do not take any measure to the infectious populations. It is necessary to study infectious disease in eco-epidemiological system by some control strategies. In this subsection, we will analyze the minimum of infective intermediate predator population by the control theory, and formulate an optimal control problem by reconsidering system (2) to reduce the numbers of infectious populations. Therefore, we introduce two control variables $u_1(t)$ and $u_2(t)$ in our system (2), where $u_1(t)$ represents efforts intended to prevent the susceptible intermediate predator from having contact with the infective intermediate predator and $u_2(t)$ represents the fraction of infective intermediate predator that will be put under treatment to reduce the number of infective intermediate predator.

Just as a coin has two sides, there will be a lot of costs generated during the control process. Our main aim is to minimize the infective intermediate predator as well as to reduce the costs required to control the disease. For this purpose, Pontryagin's

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maximum principle will be applied [Kar & Batabyal, 2011]. Thus, we form the objective function of our optimal control problem given by

$$J(u_1, u_2) = \int_0^{t_f} \left[P_2(t) + \frac{1}{2} c_1 u_1^2(t) + \frac{1}{2} c_2 u_2^2(t) \right] dt, \tag{21}$$

subject to the state system

$$\begin{cases} \dot{X} = X(r - aX) - a_1 X P_1, \\ \dot{P}_1 = e_1 a_1 X P_1 - a_2 P_1 Q_1 \\ \qquad - (1 - u_1(t)) \beta P_1 P_2 - d_1 P_1, \\ \dot{P}_2 = (1 - u_1(t)) \beta P_1 P_2 - \dfrac{a_3 P_2 Q_1}{1 + b P_2} \\ \qquad - (d_2 + c_1 + u_2(t)) P_2, \\ \dot{Q}_1 = e_2 a_2 P_1 Q_1 + \dfrac{e_3 a_3 P_2 Q_1}{1 + b P_2} - \dfrac{m a_3 P_2 Q_1}{1 + b P_2} \\ \qquad - d_3 Q_1 + u_2(t) P_2, \end{cases} \tag{22}$$

with the initial conditions

$$X(0) \geq 0, \quad P_1(0) \geq 0, \quad P_2(0) \geq 0, \quad Q_1(0) \geq 0, \tag{23}$$

where t_f is the final time and $c_1 \geq 0$, $c_2 \geq 0$ are weight factors that adjust the intensity of two different control measures. Our objective function is a continuously differentiable function of state variables and control. Pontryagin's maximum principle gives the necessary conditions to determine a feasible value of the control for which the objective function would be optimized. If this feasible control exists, then it is said to be the optimal control [Lenhart & Workman, 2007]. Here, $U = \{(u_1, u_2) \,|\, u_i(t)$ is measurable and $0 \leq u_i(t) \leq 1$, for all $t \in [0, t_f]\}$.

Next, we will investigate the existence of the optimal control of the above-mentioned problem by the results in [Fleming & Rishel, 1975].

Theorem 8. *There exists an optimal control* $u^* = (u_1^*, u_2^*)$, $t \in [0, t_f]$ *such that*

$$J(u_1^*, u_2^*) = \min_{u_1(t), u_2(t) \in U} J(u_1, u_2),$$

subject to the control system (22) with the initial conditions (23).

Proof. To prove the existence of an optimal control, we have to check the following conditions:

(1) the control and corresponding state variables are non-negative values;
(2) the control set U is convex and closed;
(3) the right side of the state system (22) is bounded by linear function in the state and control variables;
(4) the integrand of the objective functional is concave on U;
(5) there exist constants such that the integrand in (21) of the objective functional is satisfied.

In order to verify these conditions, we note that the control variable and all the state variables are non-negative. In this minimizing problem, the necessary convexity of the objective functional in $u_1(t)$, $u_2(t)$ is satisfied. Moreover, the control variable $(u_1, u_2) \in U$ is also convex and closed by definition. The optimal system is bounded, which determines the compactness. In addition, the integrand in the functional (21), $P_2(t) + \frac{1}{2} c_1 u_1^2(t) + \frac{1}{2} c_2 u_2^2(t)$ is convex on the control set U. Therefore, these conditions determine the existence of the optimal control u^* which minimizes the objective functional (21) for $t \in [0, t_f]$ with the help of the system (22). Also, we can easily see that there exist constants $b_1 > 0$, $b_2 > 0$ and $\alpha > 1$ such that

$$L(t, u_1, u_2) \geq b_1 (|u_1|^2 + |u_2|^2)^{\frac{\alpha}{2}} - b_2,$$

which completes the existence of an optimal control. ■

5.2. *The characterization of optimal control*

In order to find an optimal solution pair, we will consider the optimal control problem (21) in this subsection. We first give the Lagrangian of our optimal control problem as follows

$$L(P_2, u_1, u_2) = P_2(t) + \frac{1}{2} c_1 u_1^2(t) + \frac{1}{2} c_2 u_2^2(t). \tag{24}$$

In order to find the minimum value of the Lagrangian, we define the Hamiltonian $H = H(x(t), u_1, u_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, t)$ given by

$$\begin{aligned} &H(x(t), u_1, u_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, t) \\ &\quad = L(P_2, u_1, u_2) + \lambda_1(t)[X(r - aX) - a_1 X P_1] \end{aligned}$$

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$$
\begin{aligned}
&+\lambda_2[e_1a_1XP_1 - a_2P_1Q_1(1-u_1(t))\\
&\times \beta P_1P_2 - d_1P_1] + \lambda_3\bigg[(1-u_1(t))\beta P_1P_2\\
&-\frac{a_3P_2Q_1}{1+bP_2} - (d_2+c_1+u_2(t))P_2\bigg]\\
&+\lambda_4\bigg[e_2a_2P_1Q_1 + \frac{e_3a_3P_2Q_1}{1+bP_2} - \frac{ma_3P_2Q_1}{1+bP_2}\\
&-d_3Q_1 + u_2(t)P_2\bigg] - \omega_{11}u_1(t) - \omega_{12}(1-u_1(t))\\
&-\omega_{21}u_2(t) - \omega_{22}(1-u_2(t)),
\end{aligned}
\tag{25}
$$

where $\omega_{ij}(t) \geq 0$, $i,j = 1,2$, are the penalty multipliers satisfying

$$
\begin{aligned}
&\omega_{11}(t)u_1(t) = \omega_{12}(t)(1-u_1(t)) = 0,\\
&\omega_{21}(t)u_2(t) = \omega_{22}(t)(1-u_2(t)) = 0.
\end{aligned}
$$

At optimal control u_1^* and u_2^*, $x(t) = (X(t), P_1(t), P_2(t), Q_1(t))$ and λ_i $(i = 1,2,3,4)$ are known as the adjoint variables or the costate variables. Now we apply the necessary conditions to the Hamiltonian H in (25).

Theorem 9. *Let $X^*(t)$, $P_1^*(t)$, $P_2^*(t)$, $Q_1^*(t)$ be the solutions of the control system (22) together with the control variables $u_1^*(t)$, $u_2^*(t)$ for the optimal control problem (21), then there exist adjoint variables λ_1, λ_2, λ_3 and λ_4 that satisfy the following conditions*

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= -\lambda_1(r - 2aX - a_1P_1) - \lambda_2e_1a_1P_1,\\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= \lambda_1a_1X - \lambda_2(e_1a_1X - a_2Q_1\\
&\quad - (1-u_1(t))\beta P_2 - d_1)\\
&\quad - \lambda_3(1-u_1(t))\beta P_2 - \lambda_4e_2a_2Q_1^*,\\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}t} &= -1 + \lambda_2(1-u_1(t))\beta P_1\\
&\quad - \lambda_3\bigg[(1-u_1(t))\beta P_1\\
&\quad - \frac{a_3Q_1}{(1+bP_2)^2} - (d_2+c_1+u_2(t))\bigg]\\
&\quad - \lambda_4\left(\frac{e_3a_3Q_1}{(1+bP_2)^2} - \frac{ma_3Q_1}{(1+bP_2)^2}\right),\\
\frac{\mathrm{d}\lambda_4}{\mathrm{d}t} &= \lambda_2a_2P_1 + \lambda_3\frac{a_3P_2}{1+bP_2} - \lambda_4\bigg(e_2a_2P_1 - \frac{e_3a_3P_2}{1+bP_2}\\
&\quad - \frac{ma_3P_2}{1+bP_2} - d_3 + u_2(t)\bigg),
\end{aligned}
\tag{26}
$$

with transversality condition (or boundary condition)

$$
\lambda_i(t_f) = 0, \quad i = 1,2,3,4. \tag{27}
$$

Also, the optimal controls can be obtained from the following equations

$$
\begin{aligned}
u_1^*(t) &= \min\left\{1, \max\left\{0, \frac{1}{c_1}(\lambda_3 - \lambda_2)\beta P_1P_2\right\}\right\},\\
u_2^*(t) &= \min\left\{1, \max\left\{0, \frac{1}{c_2}(\lambda_4 - \lambda_3)\beta P_2\right\}\right\}.
\end{aligned}
\tag{28}
$$

Proof. Suppose $X(t) = X^*(t)$, $P_1(t) = P_1^*(t)$, $P_2(t) = P_2^*(t)$, and $Q_1(t) = Q_1^*(t)$. By differentiating the Hamiltonian H with respect to states, we can get the adjoint system

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= -\frac{\partial H}{\partial X}, \quad \lambda_1(t_f) = 0,\\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= -\frac{\partial H}{\partial P_1}, \quad \lambda_2(t_f) = 0,\\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}t} &= -\frac{\partial H}{\partial P_2}, \quad \lambda_3(t_f) = 0,\\
\frac{\mathrm{d}\lambda_4}{\mathrm{d}t} &= -\frac{\partial H}{\partial Q_1}, \quad \lambda_4(t_f) = 0.
\end{aligned}
\tag{29}
$$

Through the calculation, it is easy to verify the adjoint equation (26). By differentiating the Hamiltonian H with respect to the controls, we can obtain the following optimality conditions

$$
\begin{cases}
\dfrac{\partial H}{\partial u_1} = c_1u_1(t) + \lambda_2\beta P_1P_2 - \lambda_3\beta P_1P_2\\
\qquad\qquad - \omega_{11} + \omega_{12} = 0,\\
\dfrac{\partial H}{\partial u_2} = c_2u_2(t) - \lambda_3\beta P_2 - \lambda_4P_2 - \omega_{21} + \omega_{22} = 0.
\end{cases}
\tag{30}
$$

To determine an explicit expression for the optimal control without ω_1 and ω_2, a standard optimality technique is used [Fleming & Rishel, 1975]. We consider the following three cases.

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(i) If $\{t \,|\, 0 < u_1^*(t) < 1\}$, then we have $\omega_{11} = \omega_{12} = 0$. Therefore, the optimal control is

$$u_1^*(t) = \frac{1}{c_1}(\lambda_3 - \lambda_2)\beta P_1 P_2.$$

(ii) If $\{t \,|\, u_1^*(t) = 1\}$, then we have $\omega_{11} = 0$. Thus, we get

$$1 = u_1^*(t) = \frac{1}{c_1}[(\lambda_3 - \lambda_2)\beta P_1 P_2 - \omega_{12}].$$

This implies that

$$\frac{1}{c_1}(\lambda_3 - \lambda_2)\beta P_1 P_2 \geq 1, \quad \text{since } \omega_{12} \geq 0.$$

(iii) If $\{t \,|\, u_1^*(t) = 0\}$, then we have $\omega_{12} = 0$. Therefore, we have that

$$0 = u_1^*(t) = \frac{1}{c_1}[(\lambda_3 - \lambda_2)\beta P_1 P_2 + \omega_{11}].$$

This implies that

$$\frac{1}{c_1}(\lambda_3 - \lambda_2)\beta P_1 P_2 \geq 1, \quad \text{since } \omega_{11} \geq 0.$$

From the above analysis, we can obtain that the optimal control $u_1^*(t)$ is characterized as

$$u_1^*(t) = \min\left\{1, \max\left\{0, \frac{1}{c_1}(\lambda_3 - \lambda_2)\beta P_1 P_2\right\}\right\}.$$

Similarly, we can get

$$u_2^*(t) = \min\left\{1, \max\left\{0, \frac{1}{c_2}(\lambda_4 - \lambda_3)\beta P_2\right\}\right\}.$$

This completely finishes the proof. ■

6. Numerical Simulations

In order to demonstrate the theoretical results which we have obtained in the previous sections, we will give a set of values of parameters in Table 2. We also assume that the initial conditions of system (2) are $X(0) = 2.25$, $P_1(0) = 1.05$, $P_2(0) = 0.4$ and $Q_1(0) = 0.38$.

First, when we choose $r = 1.6$, $a_1 = 0.34$, $a_2 = 0.23$, $\beta = 0.28$, $e_1 = 0.28$, $e_3 = 0.25$ and $d_1 = 0.063$, we can obtain the disease-free equilibrium $E_4(1.5894, 0.0511, 0, 0.3839)$ of system (2) and $R_0 = 0.7329 < 1$. Thus, the disease-free equilibrium E_4 is globally asymptotically stable [Fig. 2(a)]. If we choose $r = 1.3$, $a_1 = 0.44$, $a_2 = 0.25$, $\beta = 0.4$, $e_1 = 0.23$, $e_3 = 0.35$ and $d_1 = 0.073$, then we can calculate that $R_0 > 1$. From Fig. 2(b), the endemic equilibrium E_1^* is globally asymptotically stable.

Table 2. Biological meaning of parameters.

Parameters	Number Value	Source
r	Variable	Variable
a	0.8	Xu and Zhang [Xu & Zhang, 2013]
a_1	Variable	Variable
a_2	Variable	Variable
a_3	0.18	Estimate
β	Variable	Variable
e_1	Variable	Variable
e_2	0.27	Estimate
e_3	Variable	Variable
d_1	Variable	Variable
d_2	0.05	Estimate
d_3	0.06	Estimate
c_1	0.25	Kant [Kant & Kumar, 2017]
m	0.3	Estimate
b	0.2	Estimate

Secondly, Fig. 3(a) shows that a forward bifurcation happens when R_0 crosses unity for system (2). A small positive asymptotically stable equilibrium appears and the disease-free equilibrium loses its stability. We choose a set of following parameters: $r = 1.9$, $a = 13.08$, $a_1 = 0.84$, $a_2 = 0.1$, $a_3 = 0.34$, $e_1 = 0.28$, $e_2 = 0.05$, $e_3 = 0.625$, $d_1 = 0.023$, $d_2 = 0.015$, $d_3 = 0.01$, $c_1 = 0.9$, $m = 0.01$, $b = 1.5$, $\beta \in \{0.539, 0.999\}$. Therefore, we obtain that $R_0 \in \{0.8093, 1.50\}$. When $R_0 < 1$, the number of infective predators reduces to zero; while $R_0 > 1$, the number of infective predators will increase or decrease to the curved line that marks the endemic equilibrium. At the same time, the positive equilibrium disappears when $R_0 < 1$. Figure 3(b) shows that a backward bifurcation takes place. We choose a set of following parameters: $a = 19.98$, $a_3 = 0.18$, $e_2 = 0.07$, $d_1 = 0.063$, $d_2 = 0.025$, $c_1 = 0.8$ and keep other parameters as the same values. When R_0 is less than unity, a small positive unstable equilibrium appears while the disease-free equilibrium and a larger positive equilibrium are locally asymptotically stable. Epidemiologically, backward bifurcation shows that it is not enough to only reduce the basic reproductive number to less than one to eliminate a disease when R_0 crosses unity.

Thirdly, we choose a set of following parameters: $r = 1.5$, $a = 0.6$, $\beta = 0.8$, $a_1 = 0.44$, $a_2 = 0.05825$, $a_3 = 0.79$, $e_1 = 0.23$, $e_2 = 0.47$, $e_3 = 4.89$, $d_1 = 0.093$, $d_2 = 0.05$, $d_3 = 0.66$, $c_1 = 0.25$,

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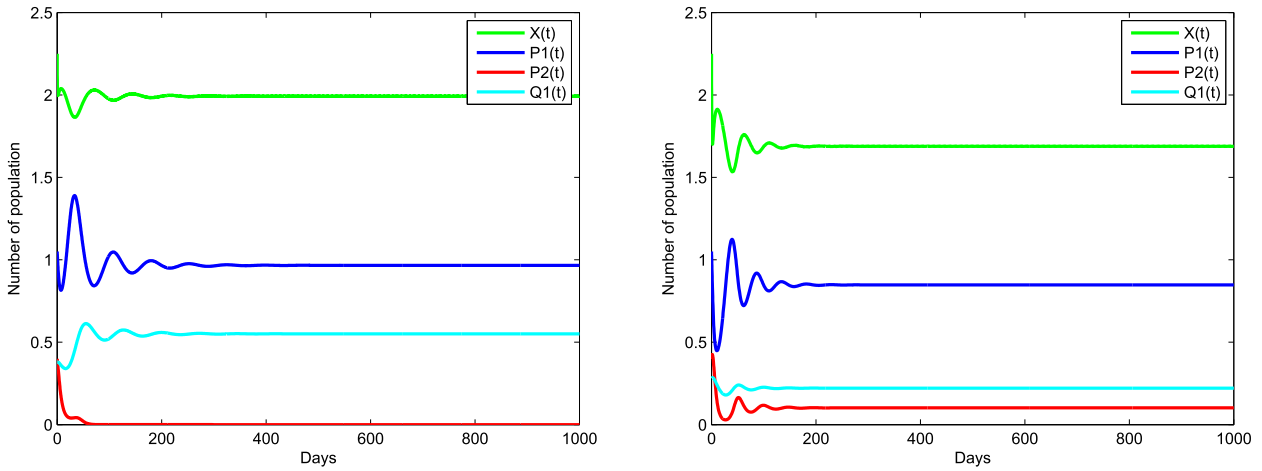

Fig. 2. System (2) is globally asymptotically stable: (a) the disease-free equilibrium E_4 and (b) the endemic equilibrium E_1^*.

$m = 0.03$, $b = 0.2$. The endemic equilibrium E_1^* of system (2) is globally asymptotically stable when $R^* > 1$ and $\beta < \beta^{**}$ (Fig. 4). From Fig. 5, we can see that the solution curves of system (2) perform a sustained periodic oscillation and phase trajectories approach limit cycle when β passes through the critical value $\beta^{**} = 0.96$. According to the theoretical results, we know that system (2) converges to a sustained periodic solution, and undergoes a Hopf bifurcation around the endemic equilibrium E_1^*.

Lastly, we investigate the optimal solutions to system (2) by numerical results. We use fourth-order Runge–Kutta forward iterative method to solve the state variables of system (2) and then solve the system (25) for the adjoint variables by backward fourth-order Runge–Kutta iterative method [Hackbusch, 1978]. Now, we select a set of following parameter values: $r = 1.8$, $a = 0.9$, $\beta = 0.5$, $a_1 = 0.8$, $a_2 = 0.59$, $a_3 = 0.49$, $e_1 = 0.35$, $e_2 = 0.135$, $e_3 = 0.137$, $d_1 = 0.4$, $d_2 = 0.1$, $d_3 = 0.9$, $c_1 = 0.36$, $m = 0.5$, $b = 0.45$. First, when $c_1 = 10$ and $c_2 = 1$, Fig. 6 indicates the changes of populations P_1 and P_2 with the different values of u_1, u_2. From Fig. 6(a), we can easily get that system (2) with control is better than that without control. When $u_1 = 0$, $u_2 = 0$, the infected predator $P_2(t)$ tends to a lowest value. When $u_1 = 0.2$, $u_2 = 0$, the number of the susceptible predators is bigger than that in the case $u_1 = 0$, $u_2 = 0.2$. Comparing with the case $u_1 = 0.2$, $u_2 = 0$, the number of susceptible predators is better in the case $u_1 = 0.2$, $u_2 = 0.2$. From Fig. 6(b), we know that the density of $P_2(t)$ decreases with the values

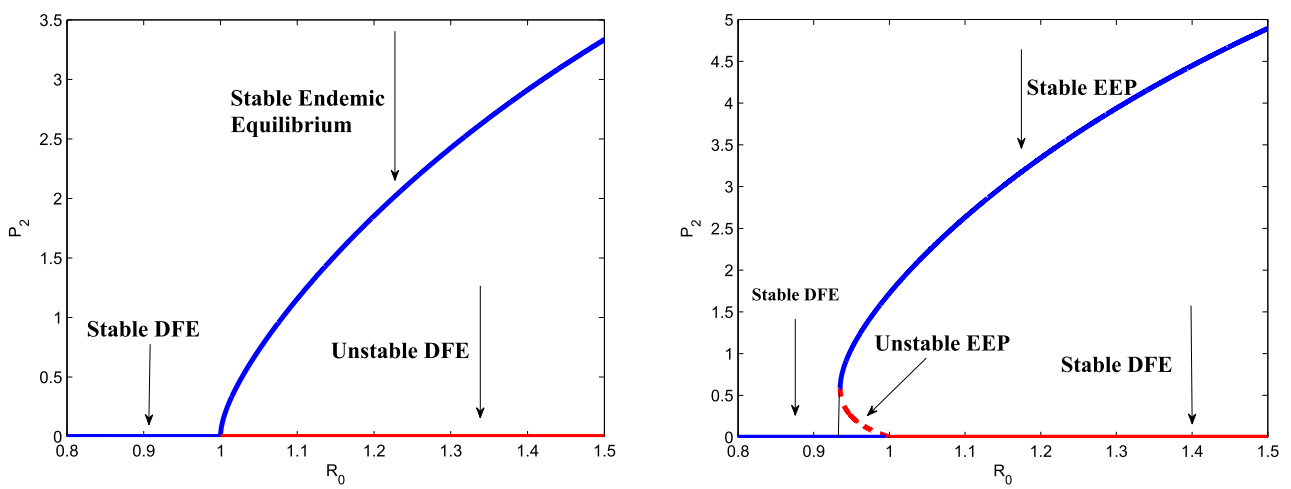

Fig. 3. Forward bifurcation or backward bifurcation takes place when R_0 crosses unity: (a) Forward bifurcation and (b) backward bifurcation.

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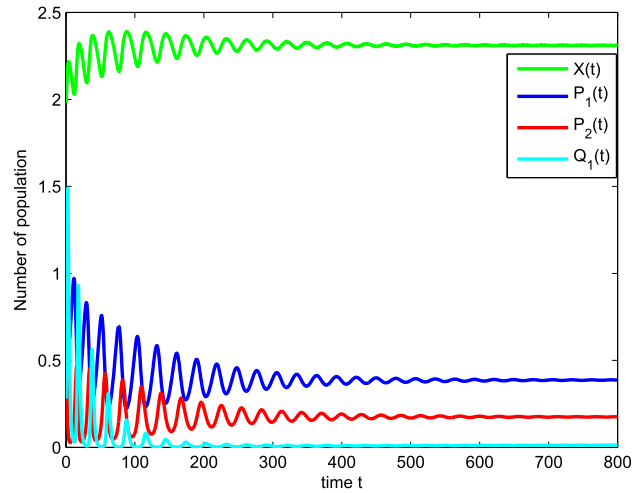
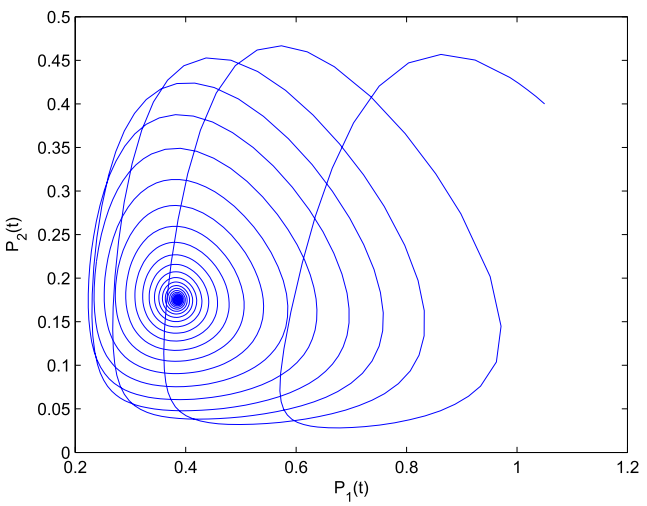
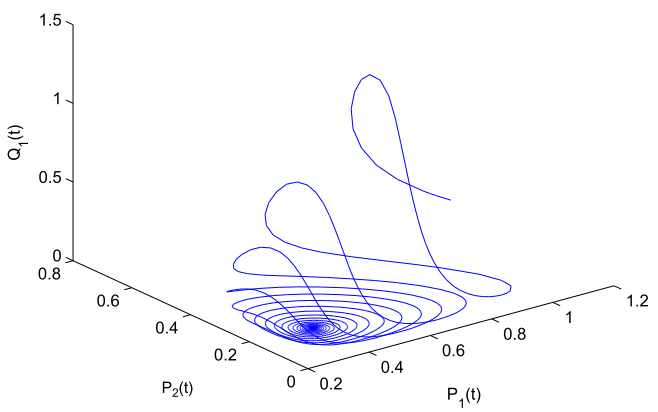


(a)

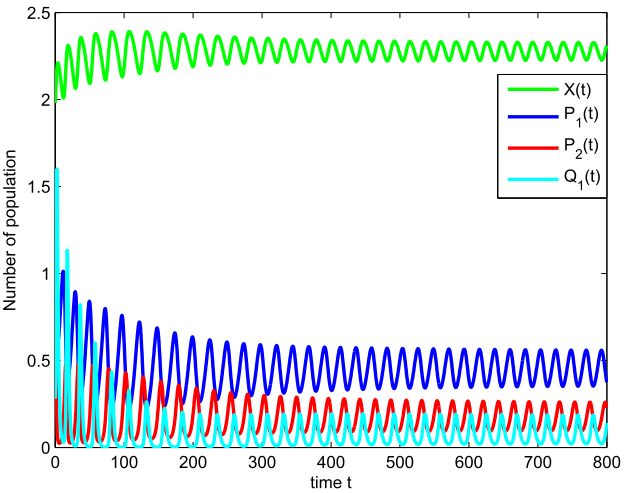


(b)

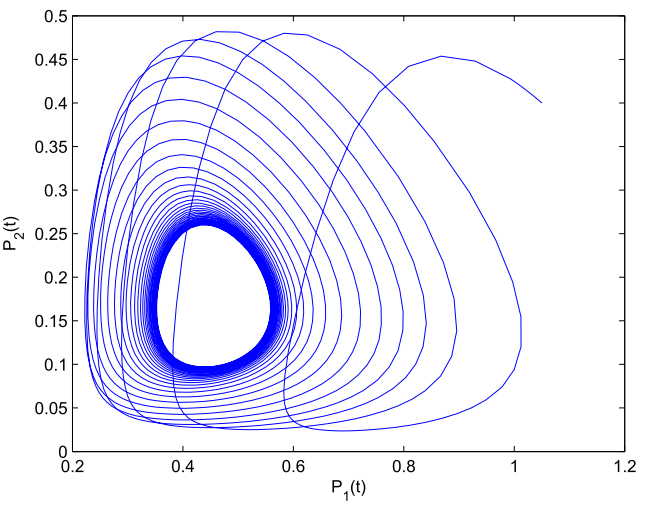


(c)

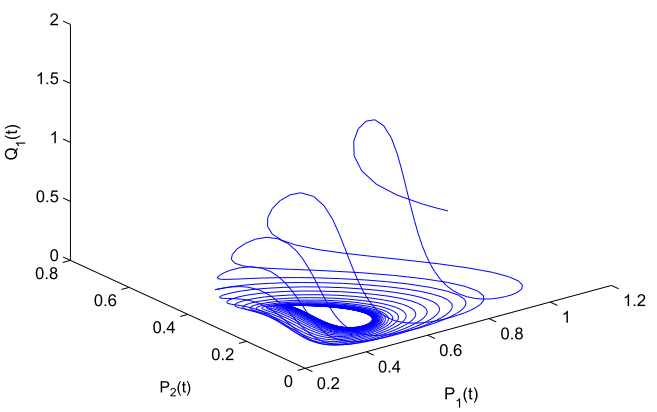
Fig. 4. The endemic equilibrium E_1^* of system (2) is globally asymptotically stable when $R_0 > 1$ and $\beta < \beta^*$: (a) Dynamic response curve, (b) phase diagram of $P_1(t), P_2(t)$ and (c) phase diagram of $P_1(t), P_2(t)$ and $Q_1(t)$.



(a)



(b)



(c)

Fig. 5. The endemic equilibrium E_1^* of system (2) is unstable when $R_0 > 1$ and $\beta > \beta^*$: (a) Dynamic response curve, (b) phase diagram of $P_1(t), P_2(t)$ and (c) phase diagram of $P_1(t), P_2(t)$ and $Q_1(t)$.

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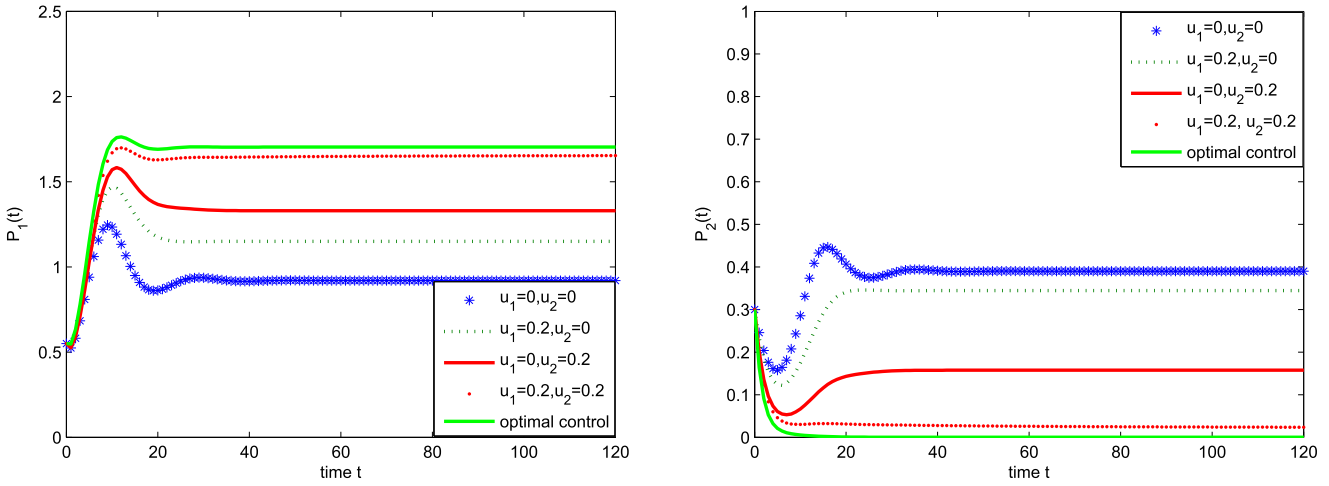
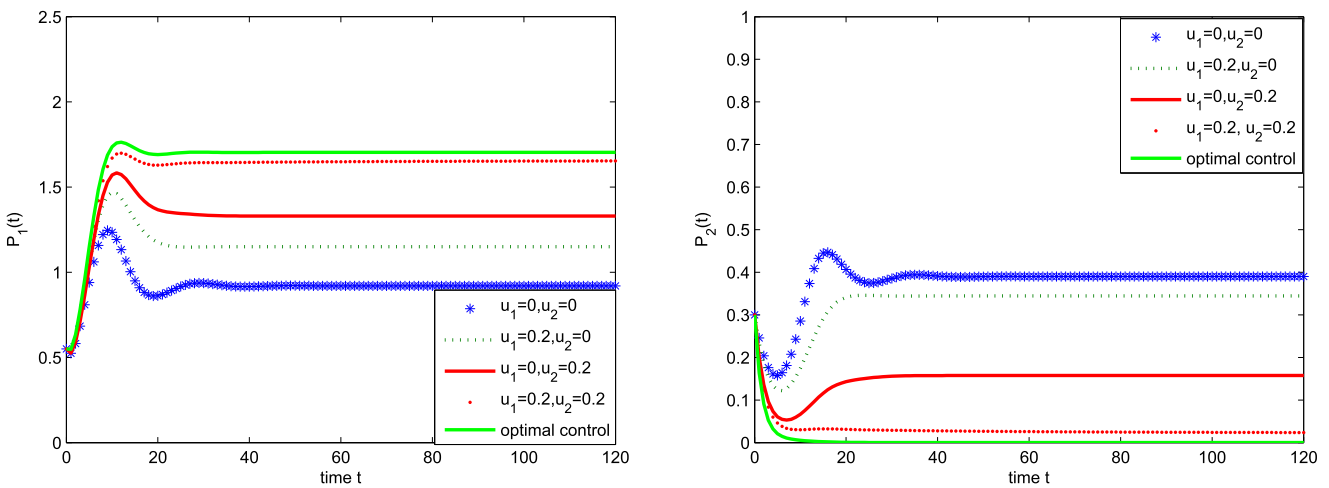


(a) (b)

Fig. 6. Variations of population of system (2) with the different values of u_1, u_2 when the weight factors in objective function are $c_1 = 10$, $c_2 = 1$: (a) the susceptible predator and (b) the infected predator.



(a) (b)

Fig. 7. Variations of population of system (2) with the different values of u_1, u_2 when the weight factors in objective function are $c_1 = 1$, $c_2 = 10$: (a) the susceptible predator and (b) the infected predator.

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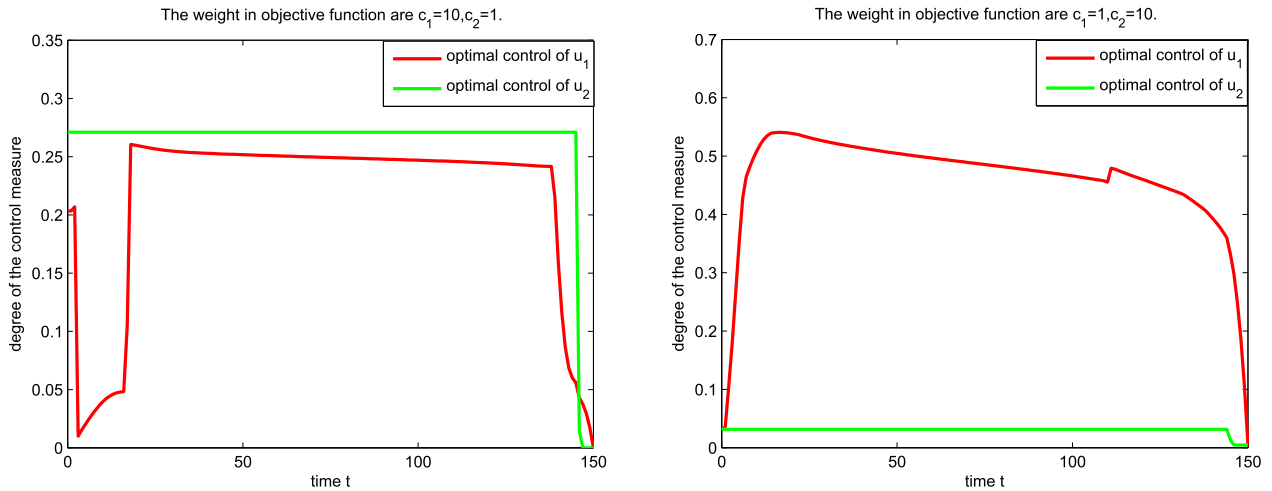

(a) (b)

Fig. 8. The impact of different weight coefficients on optimal control u_1, u_2: (a) $c_1 = 10$, $c_2 = 1$ and (b) $c_1 = 1$, $c_2 = 10$.

of u_1 and u_2 increasing. If we change weight coefficients into $c_1 = 1$, $c_2 = 10$, then we know that there is little difference between Figs. 6 and 7. So we can draw the same conclusion. Figure 8 shows the process of control measure with different weight coefficients. In Fig. 8(a), when $c_1 = 10$, $c_2 = 1$, the simulation shows that the control strength of u_2 is much longer than that of u_1. Inversely, when $c_1 = 1$, $c_2 = 10$, we can get that the control strength of u_1 is longer than that of u_2 [see Fig. 8(b)].

7. Conclusions and Discussions

In this paper, an eco-epidemiology model is established under the assumption that the disease spreads among the predator and super-predator in the ecological system. Because the susceptible super-predator can distinguish the infected predator and the susceptible predator, and catches not only the infected predator but also susceptible predator in our model. When $R_0 < 1$, there exists the disease-free equilibrium which is locally asymptotically stable. When $R_0 > 1$ and some other conditions are satisfied, the existence of the endemic equilibrium is discussed. In addition, the endemic equilibrium is also globally asymptotically stable when $R_0 > 1$. Theorem 6 shows that system (2) occurs as a backward bifurcation when $R_0 = 1$ and $R_1^* > 1$ and a forward bifurcation when $R_0 = 1$ and $R_1^* < 1$. We also prove that Hopf bifurcation takes place around the endemic equilibrium by considering the disease transmission coefficient of predator as bifurcation parameter. Furthermore, we establish the optimality system and obtain the necessary conditions of optimality by the Pontryagin's maximum principle. To observe the rich dynamical behavior, we also give some numerical simulations.

The world is full of uncertainty and random phenomena [Zhang *et al.*, 2019; Zhao *et al.*, 2020], so the species in the ecosystem may be subject to different forms of random interference. Therefore, we take the effect of fluctuations into account by stochastically perturbing the force of infection rate β of the disease, that is,

$$\beta \to \beta + \sigma \dot{B}(t),$$

where $B(t)$ is a standard Brownian motion, $\sigma^2 > 0$ is the intensity of environmental white noise. Thus system (1) can be written as

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = X(r - aX) - a_1 X P_1, \\ \dfrac{\mathrm{d}P_1}{\mathrm{d}t} = e_1 a_1 X P_1 - a_2 P_1 Q_1 - \beta P_1 P_2 - d_1 P_1 \\ \qquad\qquad - \sigma P_1(t) P_2(t) dB(t), \\ \dfrac{\mathrm{d}P_2}{\mathrm{d}t} = \beta P_1 P_2 - \dfrac{a_3 P_2 Q_1}{1 + bP_2} - (d_2 + c_1) P_2 \\ \qquad\qquad + \sigma P_1(t) P_2(t) dB(t), \\ \dfrac{\mathrm{d}Q_1}{\mathrm{d}t} = e_2 a_2 P_1 Q_1 + \dfrac{e_3 a_3 P_2 Q_1}{1 + bP_2} - \dfrac{m a_3 P_2 Q_1}{1 + bP_2} - d_3 Q_1, \\ \dfrac{\mathrm{d}Q_2}{\mathrm{d}t} = \dfrac{m a_3 P_2 Q_1}{1 + bP_2} - (d_4 + c_2) Q_2. \end{cases} \tag{31}$$

We leave these works for the future.

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Acknowledgments

The author thanks the anonymous editors and reviewers for carefully reading the manuscript and for valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (Grant Nos. 11661050 and 11861044), and the HongLiu First-class Disciplines Development Program of Lanzhou University of Technology.

Conflict of Interest

The authors declare that they have no conflict of interest relevant to the manuscript.

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