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REVIEW

Clarify the physical process for fractional dynamical systems

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Abstract Dynamics in fractional order systems has been discussed extensively for presenting a possible guidance in the field of applied mathematics and interdisciplinary science. Within hundreds and thousands of reviews, regular papers and drafts, many fractional differential equations are presented for enjoying mathematical proof without clarifying the scientific background and physical principles. It seems that all nonlinear problems on integer order systems even networks can be confirmed as fractional order systems. This mini-review gives an appropriate clarification on fractional dynamical systems from the physical viewpoint, thereby presenting sufficient evidences for further study on fractional calculus. We argued that non-uniform diffusion, boundary effect and elastic deformation account for the calculation and estimation with fractional order on some physical variables, which can be mapped into dimensionless variables in the dynamical systems. In addition, some similar definitions for energy, wave propagation and diffusion are suggested to find reliable confirmation in the application of fractional calculus.

Keywords Non-uniform diffusion · Damping force · Fractional order · Boundary effect · Memory effect

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1 Introduction

Nonlinear oscillators provide feasible bridge for theoretical analysis and prediction in nonlinear systems, which estimate the correlation between changeable and detectable variables. Furthermore, stochastic disturbance and noise can be applied on deterministic systems for possible estimation on stochastic dynamics. The occurrence and synchronization control of chaos were ever regarded as the most interesting topic in nonlinear systems, which can be described by a variety of ordinary differential equations (ODEs) with integer order. As is well known, chaotic systems can have potential applications in secure communication and image encryption [1–5], and the occurrence of chaos is an intrinsic property of biological systems [6–8]. For example, the neural activities can show quiescent, spiking, bursting and chaotic oscillation by changing the bifurcation parameters or external stimulus [9–12]. Indeed, many nonlinear electric components such as negative resistor, channel diode, Josephson junction and [13, 14], memristor [15–19] can be used to build a variety of nonlinear circuits, which can trigger distinct chaos by taming the intrinsic parameters. Furthermore, the circuit equations can be obtained according to the Kirchhoff's law, and then, an equivalent dynamical system can be approached by applying appropriate scale transformation [20] on the intrinsic parameters and variables. In fact, these circuit equations and the dimensionless dynamical system are often described by ordinary differential equations. Some of these nonlinear oscillators are used to discovery the collective behaviors in spatiotemporal systems. For example, some nonlinear oscillators and neuron models [21–25] are used to connect complex networks and neural networks [26–28] with different connection topologies and settings on boundary conditions. Also, the local kinetics of continuous media can also be described by some nonlinear oscillators, and thus, the wave propagation in the reaction–diffusion systems (RDs) [29–32] can be estimated.

Indeed, many integer order ODEs and even RDs can model and calculate the mode transition in oscillation, dynamical bifurcation, synchronization stability, wave propagation and even pattern formation in spatiotemporal systems. Some researchers claimed that fractional order dynamical systems [33–37] and fractional order network [38–40] can be more suitable for investigating nonlinear dynamics and stability of traveling waves. In particular, Wu and Baleanu [41, 42] clarified the occurrence of discrete chaos when the delayed logistic equation is discretized by utilizing the (discrete fractional calculus) DFC approach, and the discrete memory was confirmed in the discrete fractional logistic map proposed in the left Caputo discrete delta's sense. We agree that reliable theoretical models are helpful for further nonlinear analyses, but the knowledge of the underlying physical mechanism is more important and critical. For example, the authors of this review presented clear clues to recognize the physical mechanism of neurodynamics [43]. In fact, reliable physical evidences and scientific clarification are very important for understanding the application of fractional order operation. In this way, more researchers can be involved to investigate the significance of fractional order calculation on dynamical systems and diffusion in spatiotemporal systems. *Dalir* et al. [44] summarized the applications of the theory of fractional calculus and different definitions of fractional calculus were supplied for contrast. The Riemann–Liouville derivative (RLd) [45, 46] and Caputo derivative (*C* type) [47] are two most popular definitions. The RL type of fractal calculus is defined by

$$\begin{cases} {}^{\mathrm{RL}}_{\alpha}D^{\alpha}_{t}f(t) = \dfrac{1}{\Gamma(n-\alpha)}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\displaystyle\int_{a}^{t}\dfrac{f(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}, & n-1 \le \alpha < n; \\ \Gamma(z) = \displaystyle\int_{0}^{\infty} e^{-t}t^{z-1}\mathrm{d}t, & \text{Gamma function}; \end{cases} \tag{1}$$

The *C* type of fractional calculus is approached by

$$\begin{cases} {}^{C}_{a}D^{\alpha}_{t}f(t) = \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{a}^{t}\dfrac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}, & n-1<\alpha<n; \\ \left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n} f(t), & \alpha = n; \end{cases} \tag{2}$$

It is found that α order fractional derivative at time t is not defined locally, and it depends on the total effects of the commonly used n-order integer derivative on the interval $[a, t]$. Therefore, it can describe the variation of a system in which the instantaneous change rate depends on the past state, which is called the memory effect in a visualized manner.

Among the known three kinds of fractional calculus [48], the Grünwald–Letnikov (GL) type [44, 48, 49] is a kind of joined definition; it calculates as follows

$$ {}^{\mathrm{GL}}_{\alpha}D^{\alpha}_{t}f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{\left[\frac{t-\alpha}{h}\right]}(-1)^{j}\binom{\alpha}{j}f(t-jh); \tag{3}$$

This kind of limit calculus was ever enjoyed in discretization operation. Indeed, Ortigueira et al. [48] suggested that Liouville, Riemann–Liouville and Caputo derivatives are extended to the complex functions space, and they established a bridge with existing integral formulations and obtained regularized integrals for the three types of fractional calculus when it was started from a complex formulation of the Grünwald–Letnikov derivative. Garrappa [49] provided an explicit representation in terms of fractional differences of Grunwald–Letnikov type, and operators for the representation in the time-domain of systems with relaxation of Havriliak–Negami type were studied. Tarasov [50] confirmed that linear and nonlinear equations with the Caputo–Fabrizio operators can be represented as systems of differential equations with derivatives of integer orders, but the Caputo–Fabrizio operators with exponential kernel cannot describe non-locality and memory (temporal non-locality) in processes and systems. Compared with the classical fractional derivatives, Ortigueira et al. [51] concluded that Caputo–Fabrizio (CF) and Atangana–Baleanu (AB) operators (definition) perform poorly when two alternative models based on the CF and AB operators are assessed to match with datasets obtained from electrochemical capacitors and the human body electrical impedance. Ortigueira and Machado [52] described a framework for compatible integer and fractional derivatives/integrals in signals and systems context, and it is confirmed that suitable fractional formulations are really extensions of the integer order definitions currently used in signal processing. In particular, it gives guidance to tickle the initial conditions. Indeed, Tarasov [53] proposed a principle of non-locality for fractional derivatives and confirmed that if the differential equation with fractional derivative can be presented as a differential equation with a finite number of integer order derivatives, then this fractional derivative cannot be considered as a derivative of non-integer order. As a result, the M-fractional derivative, the alternative fractional derivative, the local fractional derivative and the Caputo–Fabrizio fractional derivatives with exponential kernels are not suitable to handle with fractional derivatives of non-integer orders [53]. Gu et al. [54] proposed a new class of fractional differential equations with the Riesz–Caputo derivative, and the boundary value problem is investigated under some conditions with clear physical meaning. Wu et al. [55] investigated the finite-time stability of Caputo delta fractional difference equations, and a finite-time stability criterion was proposed for fractional differential equations and discrete fractional case.

In Ref [56]., it is confirmed that the fractional model perfectly fits the test data of memory phenomena in different disciplines by using numerical least square method. That is, a physical meaning of the fractional order is an index of memory. Without considering the memory effect, Karci [57] interpreted the geometrical meaning of the fractional order operators of any function in the case of very small value of h, in which distance, velocity and acceleration were used to depict interpretations. The specific calculation term without memory is defined by

$$f^{(\alpha)}(x) = \lim_{h\to 0}\frac{f^{\alpha}(x+h) - f^{\alpha}(x)}{(x+h)^{\alpha} - x^{\alpha}} = \left[\frac{f(x)}{x}\right]^{\alpha-1}\frac{\mathrm{d}f(x)}{\mathrm{d}x}; \tag{4}$$

As a consequence, the result of derivative process is different from classical derivative and applied operator is nonlinear when the order of fractional order derivative is not equal to 1. Only when the fractional order $\alpha = 1$, the result of derivative process approaches to classical derivative and operator is linear. For example, the equation of distance for moving object $X(t) = V_0t + 2t^2$, and the velocity is accelerated from V_0 with a constant acceleration 4; according to Eq. (2), the velocity can be obtained by $V^{(\alpha)}(t) = [V_0 + 2t]^{\alpha-1}(V_0 + 4t)$. When the acceleration is time-varying as $3t$ and the distance for moving object is $X(t) = V_0t + 0.5t^3$, then the velocity involved in fractional order definition can be obtained by $V^{(\alpha)}(t) = [V_0 + 0.5t^2]^{\alpha-1}(V_0 + 1.5t^2)$. In fact, the calculation is approached with integer order when the derivative is equal to 1 while the history information could be missed within this definition. One interesting question is when factional order derivative can be used to model and calculate the dynamics in complex systems [58]. The most favorite definition for fractional order derivative can consider the correlation of variables in time, and thus, most of the initial-dependent dynamical problems [59–61] can be estimated by using fractional order calculation. On the other hand, some researchers suggested some physical and geometrical interpretation of fractional operators and approach with fractional order derivative [62–66] by supplying specific examples. Furthermore, the reliability of algorithm for fractional order calculation was discussed [67–69] for ensuring higher accuracy and shorter calculating period. Indeed, it is important to clarify when fractional order derivative should be applied for estimating the nonlinear correlation in variable and dependence on variables. In our opinion, in case of distinct memory effect, stochastic diffusion, energy leakage, irregular boundary effect, spatial heterogeneity, fractional order derivative can be applied to discern the nonlinearity between variables. In the following section, several examples are provided to explain the fractional order calculation on physics relevant problems.

2 Non-uniform damping force on moving particles

Viscous and frictional resistance play an important role in changing the gait of moving microscopic particles and macroscopic objects. As an example, a simple oscillator in a viscous liquid, the diagram is shown in Fig. 1.

As is well known, for a smooth spherical matter particle m with radius a moving into a viscous fluid [70, 71], the viscous force is estimated as follows

$$f = 6\pi\eta ua = -k_0 u; \tag{5}$$

where η *and* u represent the viscosity coefficient of the liquid and moving velocity of particle (oscillator), respectively. The symbol "$-$" indicates the direction of this drag force. According to Bernoulli principle [72, 73], it obtains

$$P_0 + \frac{1}{2}\rho u^2 + \rho gh = C_{\text{constant}}; \tag{6}$$

where P_0 is the intensity of pressure in the of liquid (stayed with the particle m), ρ is the liquid density, g is the acceleration of gravity, h denotes the height of the particle position and C_{constant} is a constant. It confirms that the sum of kinetic energy, gravitational potential energy and pressure potential energy are kept as constants. Therefore, it is helpful to estimate the stress distribution of moving particle in the liquid under the law of conservation of mechanical energy. As a result, the time-varying pressure (drag force) f for the particle can be estimated by multiplying intensity of pressure P_0 and the superficial area S of this particle on the Eq. (6); it gets as follows

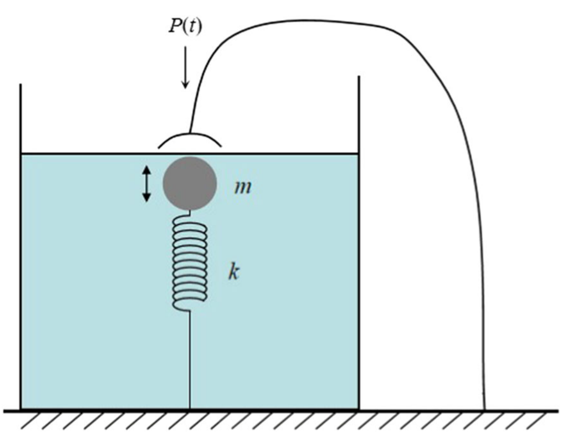

Fig. 1 An oscillator driven by periodical stimulus immersed into a viscous liquid with a viscosity coefficient η. $P(t)$ denotes the continuous stimulus acting on the matter particle m and the elasticity of spring is presented with k. The spring represents a mechanical spring immersed into the fluid, and it is used to connect the particle for generating nonlinear oscillation

$$\begin{cases} f = P_0 S \approx 4\pi a^2 P_0 = 4\pi a^2 \left(C - \rho gh - \frac{1}{2}\rho u^2 \right) \\ \quad = -k_1(\rho)u^2 + \Delta C_{\text{constant}}; \\ k_1(\rho) = 2\pi a^2, \quad \Delta C_{\text{constant}} = 4\pi a^2 (C - \rho gh); \end{cases} \tag{7}$$

That is, the dependence of pressure on the moving particle on the velocity is nonlinear and the nonlinearity is enhanced when the liquid is anisotropic, which the liquid density ρ within spatiotemporal distribution. To be consistent with Eq. (5), Eq. (7) can be approached with Eq. (5) when the particle is moving in slight, but tiny velocity and the term u^2 are approached by using Taylor series expansion as follows

$$\begin{aligned} f(u) &= f(u_0) + f'(u_0)(u - u_0) + O(u) + \Delta C_{\text{constant}} \\ &\approx -k_0 u + \xi(t); \end{aligned} \tag{8}$$

where $\xi(t)$ can be thought as stochastic disturbance. Indeed, the pressure of the moving particle is dependent on the velocity of this particle and the deformation size of this particle should be considered. Therefore, scale factor s (or q in some references) can be involved into the relation between moving velocity and pressure as follows

$$f = -k' u^s; \tag{9}$$

where the scale factor s is dependent of the deformation degree of the particle size. From mathematical viewpoint, Eq. (9) with appropriate scale factor s can be equivalent to Eqs. (8), (7) and (5) by applying the same Taylor series expansion.

Therefore, the motion equation of this particle and spring oscillator in Fig. 1 can be obtained by

$$\begin{cases} \frac{dx}{dt} = u; \\ m\frac{du}{dt} = -k' u^s + P(t) + \xi(t); \end{cases} \tag{10}$$

As a result, fractional order dynamical system can be proposed for investigating the dynamical problems when the moving particle is controlled by the viscous and frictional blocking, which the scale factor s is selected with fractional number. Similar to the memristive network [74–76], in which the local kinetics is described by a memristive oscillator that can be mapped and developed from a nonlinear circuit involved with memristor, a variety of fractional order dynamical systems can also be proposed to investigate the dynamics of more moving particles in the liquid due to the activation of viscous and frictional driving and blocking. From the viewpoint control, the viscous pressure on the particle just introduces a kind of differential control from Eq. (10), and the similar control mechanism can be confirmed in the nonlinear circuits coupled by a capacitor [77, 78] which can trigger time-varying electric field in the coupling component for energy pumping.

3 Field coupling and boundary effect

In the last decades, resistor-based voltage coupling has been used for stabilizing synchronization between chaotic circuits. Furthermore, this kind of direct variable coupling is often used to investigate collective behaviors of networks composed of generic nonlinear oscillators. As confirmed in the review [20], a variety of electric components can be used to build nonlinear circuits and also be effective to connect the output ends of nonlinear circuits by activating different kinds of coupling channels. For example, when a capacitor is used to couple two nonlinear circuits, the coupling capacitor is charged and discharged for generating time-varying current in the coupling channel and energy pumping is activated to balance the output voltages [78, 79]. On the other hand, the involvement of induction coil [80, 81] in the coupling channel can trigger time-varying magnetic field for energy pumping; thus, the output voltage can be balanced for reaching possible synchronization. Both kinds of field coupling can pump energy from the coupled circuits while no additive Joule heat is consumed and then synchronization can be realized completely. In fact, these ideal coupling components seldom consider the boundary effect which the ports of electric components can emit electromagnetic wave and a part of field energy is released. As a result, when capacitor and induction coil are combined for designing artificial hybrid synapses [82], the effect of energy radiation and release becomes distinct and important. The diagram for this kind of field coupling via capacitor and induction coil is shown in Fig. 2.

For simplicity, the nonlinear circuits A and B in Fig. 2 are composed of one capacitor, one inductor and appropriate nonlinear resistors. According to the physical Kirchhoff's law, the coupled circuits in the presence of field coupling can be, respectively, estimated by

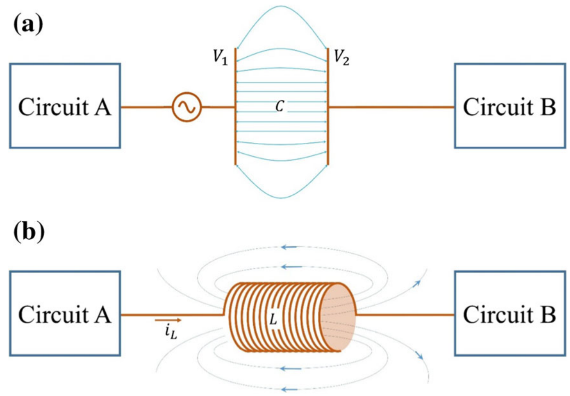

Fig. 2 Schematic diagram for nonlinear circuits under field coupling is plotted by connecting the output ends with a capacitor and/or induction coil for **a** electric field coupling and **b** magnetic field coupling. The red and black direction curve with arrow indicates the electric fluxline in coupling capacitor (and magnetic curve in the coupling indicator) embedded in the coupling channel. V_1, and V_2 represent the output voltage from the end of each nonlinear circuit, respectively. i_L denotes the induction current across the coupling coil when the coupling channel is activated

$$\begin{cases} C_1\dfrac{dV_1}{dt}=f(V_1,\ I_1)-i_c; \\ L_1\dfrac{dI_1}{dt}=g(V_1,\ I_1); \\ C_2\dfrac{dV_2}{dt}=f(V_2,\ I_2)+i_c; \text{capacitor coupling}; \\ L_2\dfrac{dI_2}{dt}=g(V_2,\ I_2); \\ i_c=C\dfrac{d}{dt}(V_1-V_2); \end{cases} \tag{11}$$

$$\begin{cases} C_1\dfrac{dV_1}{dt}=f(V_1,\ I_1)-i_L; \\ L_1\dfrac{dI_1}{dt}=g(V_1,\ I_1); \\ C_2\dfrac{dV_2}{dt}=f(V_2,\ I_2)+i_L; \text{inductor coupling}; \\ L_2\dfrac{dI_2}{dt}=g(V_2,\ I_2); \\ L\dfrac{di_L}{dt}=V_1-V_2; \end{cases} \tag{12}$$

where L_1, C_1 and L_2, C_2 represent the inductance and capacitance of electronic components of the coupled circuits, respectively. C (and L) denotes the capacitance (and inductance of coupling coil) of coupling capacitor. Due to boundary effect, the distribution of charges on the polar plates could be non-uniform, and the authors advise that the energy in the coupling capacitor can be possibly estimated by

$$E_C = \frac{1}{2}C(V_1 - V_2)^s; \quad (13)$$

where the parameter s is considered as scale factor with handling the transition of electric field distribution because some field energy is released and emitted from the boundary of the coupling component. On the other hand, it is also acceptable to estimate the electric field energy with $E_c = 0.5C^s(V_1 - V_2)^2$ and the scale factor s is associated with the property of the capacitor as well. In fact, the ideal gas equation estimates the relation between air pressure P and gas volume V with PV = constant, while adiabatic process and polytropic process can be estimated by PV^γ = constant and PV^n= constant, respectively. The scale factor γ (= C_p/C_v) is estimated by the ratio of isobaric molar heat capacity C_p and equimolar heat capacity C_v. Considering the physical units, PV^γ = constant holds the same dimension as physical energy. Inspired by this, we proposed the general energy assumption for energy in the capacitor as shown in Eq. (13). By the same way, the authors suggested that the energy pumping in the coupling induction coil and channel can be estimated by

$$E_L = \frac{1}{2}Li_L^s = \frac{1}{2L}[\int (V_1 - V_2)\mathrm{d}t]^s; \quad (14)$$

That is, the integer differential and integral calculation are not suitable to approach the exact energy pumping and mutation and revulsion (sudden change) in the density of distribution of charges on polar plates and current density across the coupling induction coil. In fact, radiation and leakage of magnetic field energy become inevitable when time-varying current pass across the induction coil, it is also acceptable to estimate the field energy as $E_L = 0.5L^s i_L^2$, and the scale factor s is dependent on the physical property of the induction coil. Therefore, similar definition for the induction current across the coupling capacitor can be suggested by

$$\begin{cases} i_c = \dfrac{\mathrm{d}^s \Delta q}{\mathrm{d}t^s} = C\dfrac{\mathrm{d}^s(V_1 - V_2)}{\mathrm{d}t^s}; \\ L\dfrac{\mathrm{d}^s i_L}{\mathrm{d}t^s} = V_1 - V_2; \end{cases} \quad (15)$$

where the scale factor s is a fractional value and the relation between voltage and charged current (induction current) can be estimated by Caputo derivative when reliable algorithm is applied. In fact, the current and voltage defined in Eq. (15) are often obtained from one branch of the nonlinear circuits, and thus, getting exact solutions becomes difficult except turning to numerical approaches. When these physical variables and parameters are mapped into dimensionless variables and bifurcation parameters by applying standard scale transformation, the effect of boundary relative sudden change in physical field distribution can be estimated with gradient current by applying fractional order calculation on the charge flux.

4 Non-uniform diffusion in spatiotemporal system

In realistic spatiotemporal systems, heat source, energy source and signal source can emit energy flow and the media can be activated for possible propagation of pulse and wave fronts. For example, the sinoatrial node in the heart can generate and send out continuous electric signal for generating stable target wave in the cardiac tissue, and this kind of electric signal can adjust the release and pumping of calcium for activating the systolic and diastolic function of heart. However, the cardiac tissue is anisotropic, and thus, the diffusion becomes non-uniform in space and estimation of spatial parameters distribution becomes difficult when reaction–diffusion equations are proposed with integer order calculation. On the other hand, the heat conduction may encounter the same heterogeneity blocking in realistic media and the integer order calculation on spatiotemporal distribution of heat should be improved for estimating the sudden changes in gradient temperature when heats are propagated in the media. This kind of non-uniform diffusion of heat and wave in the media is plotted in Fig. 3.

The developed spatial patterns can be reproduced in most of the reaction–diffusion systems by activating stochastic diffusion. For example, diffusive poisoning in the ion channels in the neural network composed of biological neurons can suppress the electric firing, and channel blocking-induced defects crack the synchronous firing asymmetrically no matter which boundary condition is applied. When finite size is considered, no-flux boundary condition is often applied while periodical boundary condition is often selected in spatiotemporal systems with large or infinite size. In fact, the diffusion and development of spatial patterns are dependent on continuous supply and propagation of energy. In case of heterogeneity, energy propagation and transmission become non-uniform. To confirm and predict this asymmetric diffusion in integer order spatiotemporal systems, stochastic disturbance becomes important while the algorithm for boundary condition for heterogeneity shows much difficult. From the viewpoint of numerical calculation, spatial distribution for diffusion coefficient can be carefully selected to stand for the standard Laplace operator in reaction–diffusion systems, while how to confirm the spatial diffusion coefficients becomes unknown. In ecological systems, the species distribution determinates the hunting range of different predators. On the other hand, stochastic hunting from predators can change the distribution of species and preys. It becomes difficult to estimate and predict the hunting range pattern within spatiotemporal systems with integer order calculation. Heat propagation in materials can be handled as pattern formation, and the diffusion equation in two-dimensional space can be obtained by

$$\rho c_p \frac{\partial T}{\partial t} = q + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right); \tag{16}$$

where ρ is material density, q represents the density of heat flow, k is thermal conductivity and c_p denotes the specific heat capacity at constant pressure, respectively. In realistic media, the thermal conductivity can be in spatiotemporal distribution because the physical property is adjusted when the material is accumulated with heat within transient period. For equivalent approach, fractional order Laplace operator can be suggested for potential application in estimating the distribution of heat conduction as follows

$$\rho c_p \frac{\partial T}{\partial t} = q + k\frac{\partial^s T}{\partial x^s} + k\frac{\partial^s T}{\partial y^s}; \tag{17}$$

where Riesz space [83] derivative is applied to describe the gradient effect of temperature. The fractional Laplace operators are collected and compared in the survey [84]. In addition, Caputo derivative can also be involved to reconsider the evolution of temperature with time. For similar pattern formation and wave propagation in the reaction–diffusion systems and spatial networks, fractional order calculation can also be applied to consider the effect of non-uniform diffusion [85], and these equations can be calculated in similar definition as follows

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^s u; \\ \frac{\partial v}{\partial t} = g(u,v); \end{cases} \tag{18}$$

$$\begin{cases} \frac{d^s u_{ij}}{dt^s} = f(u_{ij}, v_{ij}) + D\sum_{m=1,n=1}^{N} \varepsilon_{ijmn} u_{mn}; \\ \frac{d^s v_{ij}}{dt^s} = g(u_{ij}, v_{ij}); \end{cases} \tag{19}$$

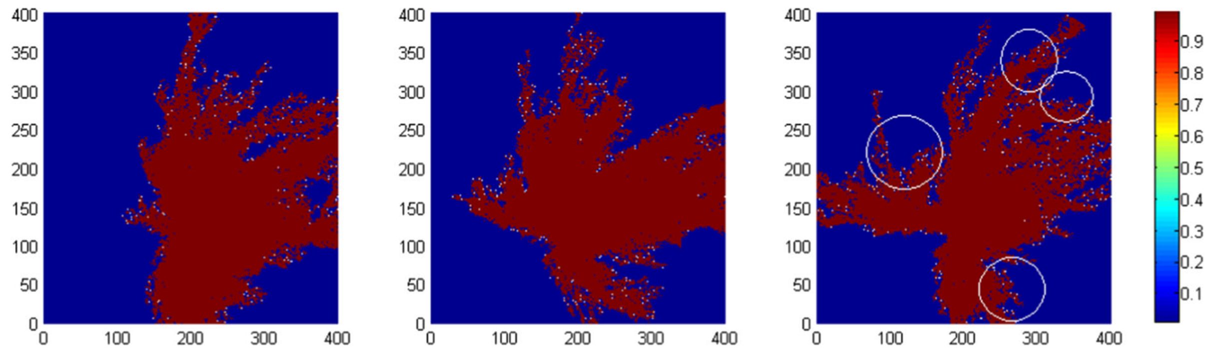

Fig. 3 Non-uniform diffusion in heat and wave in the media with heterogeneity, and the brownish red area represents the wave and energy propagation while the rest region keeps intact without invasion. The area marked with a white circle depicts the area supporting the spiral wave, and this kind of non-uniform diffusion of spatial patterns can be reproduced in excitable neural network by blocking the ion channels, which can generate heterogeneity in the media as well. Snapshots are plotted in color scale

where D_u represents the diffusion coefficient in the reaction–diffusion system shown in Eq. (18) and the Riesz space derivative can be adopted to consider the complexity in spatial diffusion. D in Eq. (19) denotes the coupling intensity and the connection matrix ε_{ijmn}= 1 when the node *ij* is connected to the node *mn,* otherwise, ε_{ijmn}= 0. Here, Caputo derivative can be used to calculate the temporal evolution of variables on any nodes as well. As a result, appropriate setting for the scale factor s can be suitable to estimate the wave propagation and heat distribution in media completely under non-uniform diffusion.

Up to date, a variety of fractional order dynamical systems have been proposed to estimate dynamics in many complex systems while some of the works seldom clarified the biophysical evidences and sufficient scientific background. In fact, most of the relevant discussions are focused on finding mathematical solutions, proof for existence of stability and producing similar results within integer order dynamical systems. Therefore, many researchers believed that the same schemes and scientific questions can be reproduced and discussed in the fractional order systems by replacing integer order with fractional order calculation. For example, many fractional order neuron models and fractional order circuits [86–96] are proposed to show the response in neural activities and mode transition under external stimulus. Furthermore, these fractional order neuron models and nonlinear circuits can be further used for building fractional order networks. What is the potential advantage for proposing fractional order neurons? In our opinion, fractional order neuron models should fit with the self-adaption and memory effect of biological neurons, and biophysical effect should be considered during the activation of any firing modes. For example, it is important to estimate the mode transition in neural activities intermittently when electromagnetic radiation is applied on the neurons. In fact, the most important reason could be that biological neurons are elastic and the synapses have fractional distribution in anatomical structure and the biophysical field intracellular and extracellular is non-uniform.

Up to date, many dynamical systems and models with integer order have been extended to calculate the dynamics in fractional order type. From the mathematical viewpoint, it is acceptable to replace the integer order with fractional order on most of the dynamical systems, and stability analysis can be further applied. However, reliable computational models should take account into the physical principle and scientific background (ecological, biological and engineering evidences) and fast effective algorithm are also critical. *Machado* et al. [97–99] presented new perspective for definitions of fractional derivatives, and rich examples were supplied for extensive investigation on the application of fractional calculus. The nonlinear systems show complex dynamics, in which this history information is rich and the forthcoming information becomes difficult for possible prediction. When the three classical fractional calculus are used to model and potential optimal control, the reliability of numerical algorithm becomes very important. When memristor is involved to build memristive circuits, the dynamics of the system is much dependent on the initial value of memristive variable (e.g., magnetic flux). Petras [100] ever discussed numerical application in chaotic behavior analysis of fractional memristor-based systems. However, the accumulation of numerical error becomes significant and numerical methods of higher accuracy should be reconsidered. The predictor–corrector approach [101] is one of the most often used methods for fractional differential equations, and it provides a simulation tool for fractional modeling accurately. For extensive guidance, readers can find possible help in the instructive works [102]; thus, the most suitable scheme can be selected for more specific fractional order systems. As is well known, the reliability of discretization is critical for handling continuous dynamical systems. Therefore, some researchers appreciated the dynamics in maps and discrete systems because some of them are more effective in processing digital signals in experiments. By the same way, discrete fractional calculus was recently proposed on time scale theory [103, 104], and it can provide an exact discretization method which leads to less numerical errors caused by the memory effects. Some recent applications in fractional discrete-time systems [41, 105] exhibit the new feature.

In a summary, it is important to find more evidences for the application of fractional calculation when the potential mechanisms have been clarified. Many researchers with reliable knowledge in applied mathematics can optimize the algorithm while the definition should depend on the physical mechanism and relevant background condition. Many readers need clear interpretation when fractional calculus is applied to solve scientific problems. This mini-review presents some suggestions and similar definition for nonlinear science; it indicates which condition can find reliable solution by applying fractional calculation. The boundary condition, initial setting, non-uniform diffusion and viscoelasticity all indicate the complex effect of memory in these nonlinear processing. On the other hand, the effect of fractional calculation can provide helpful guidance for improving the physical properties of electronic components and propose feasible strategy for economic field. For example, the stock markets are characterized by long-range correlations and persistent memory [106], which are found in natural and artificial systems, and these features are well modeled by means of the tools of fractional calculus. As is well known, the physical memristor has distinct memory effect, and the application of fractional calculus in this component becomes much attractive [107, 108], in which multidimensional scaling locus and fractional generalization of memristor were investigated in detail.

5 Conclusions

In this review, several examples relevant to physics problems are supplied to explain why fractional order calculation should be applied on nonlinear oscillators and spatiotemporal systems. Damping and viscoelastic deformation induces uniform distribution in pressure on the moving particles, boundary and memory effect in the electric components involved in the active nonlinear circuits, heterogeneity and non-uniform diffusion in the spatiotemporal system, e.g., asymmetric diffusion of oxygen in alveoli, electric signal propagation in cardiac tissue, proposed challengeable questions for the deterministic systems with integer order calculation due to time-varying spatial distribution in intrinsic parameters. Therefore, fractional order calculation can be applied to approach exact solution when these intrinsic parameters are supposed with invariant values. In the last twenty years, many fractional order systems are used to discuss the dynamics, wave stability, initials and boundary effect while the scientific background of these nonlinear equations with fractional order operator is left out. From dynamical viewpoint, distributed time delays and stochastic disturbance can be introduced in the regular systems, which can be further tamed to reproduce similar dynamics in fractional order dynamical systems. In fact, the main losses are physical memory effect and possible energy leakage in the boundary when a system is described by regular derivative. For obtaining exact modeling and calculation, fractional order derivative should be applied on complex systems with distinct memory effect, non-uniform diffusion, energy leakage, boundary effect, viscoelastic force. Our mini-review does not discuss exact algorithm and solutions for these fractional order systems while we just want to supply some possible evidences and background knowledges for readers in this field.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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